AN INTRODUCTION TO ELECTRODYNAMICS

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FROM THE STANDPOINT OF THE ELECTRON THEORY

BY

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PREFACE

The object of this book is to present a logical development of electromagnetic theory founded upon the principle of relativity. So far as the author is aware, the universal procedure has been to base the electrodynamic equations on the experimental conclusions of Coulomb, Ampère, and Faraday, even books on the principle of relativity going no farther than to show that these equations are covariant for the Lorentz-Einstein transformation. As the dependence of electromagnetism on the relativity principle is far more intimate than is suggested by this covariance, it has seemed more logical to derive the electrodynamic equations directly from this principle.

The analysis necessary for the development of the theory has been much simplified by the use of Gibbs' vector notation. While it is difficult for those familiar with the many conveniences of this notation to understand why it has not come into universal use among physicists, the belief that some readers might not be conversant with the symbols employed has led to the presentation in the Introduction of those elements of vector analysis which are made use of farther on in the text.

Chapter I contains a brief account of the principle of relativity. In the second chapter the retarded equations of the field of a point charge are derived from this principle, and in Chapter III the simultaneous field of a moving charge is discussed in some detail. In the next chapter the dynamical equation of the electron is obtained, and in Chapter V the general field equations are derived. Chapter VI takes up the radiation of energy from electrons, and Chapters VII and VIII contain some applications of the electromagnetic equations to material media, chosen as much for their illustration of the theory as for their fundamental importance. Throughout, great pains

have been taken to distinguish between definitions and assumptions, and to carry on the *physical* reasoning as rigorously as possible. It is hoped that the book may be found useful by those lecturers and students of electrodynamics who are looking for a logical rather than a historical account of the science. The subject matter covers topics appropriate for a one-year graduate course in electrodynamics and electromagnetic theory of light.

The author wishes to acknowledge his debt to those great thinkers, Maxwell, Poynting, Gibbs, Lorentz, Larmor, and Einstein, and to express his appreciation of the inspiration and unfailing interest of his former teacher, Professor H. A. Bumstead. His thanks are due his colleague, Professor H. S. Uhler, for many suggestions tending toward greater clearness of exposition.

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INTRODUCTION

ELEMENTS OF VECTOR ANALYSIS

Addition and multiplication. A *vector* is defined as a quantity which has both magnitude and direction. It will be designated by a letter in **blackface** type, its scalar magnitude being represented by the same letter in *italics*. Geometrically, a vector may be represented by an arrow having the direction of the vector and a length proportional to its magnitude. The beginning of this representative straight line is known as its *origin*, and the end, as its *terminus*. To add two vectors $\mathbf{P}$ and $\mathbf{Q}$ place the origin of $\mathbf{Q}$ at the terminus of $\mathbf{P}$. Then the line drawn from the origin of $\mathbf{P}$ to the terminus of $\mathbf{Q}$ is defined as the sum of $\mathbf{P}$ and $\mathbf{Q}$. To subtract $\mathbf{Q}$ from $\mathbf{P}$ reverse the direction of $\mathbf{Q}$ and add. The *components* of a vector are any vectors whose sum is equal to the original vector. Although, strictly speaking, the components of a vector are themselves vectors, the term component will often be used to denote the magnitude alone in cases where the direction has already been specified.

A vector is often determined by its components along three mutually perpendicular axes X, Y, Z. These axes will always be taken so as to constitute a right-handed set; that is, so that a right-handed screw parallel to the Z axis will advance along this axis when rotated from the X to the Y axis through the right angle between them. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors parallel

respectively to the X, Y, Z axes. Then if the projections of $\mathbf{P}$ along these axes are denoted by P_x, P_y, P_z,

$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}, \tag{1}$$

and, obviously,

$$\mathbf{P} + \mathbf{Q} = (P_x + Q_x)\mathbf{i} + (P_y + Q_y)\mathbf{j} + (P_z + Q_z)\mathbf{k}. \tag{2}$$

If two or more vectors are parallel to the same straight line, they are said to be *collinear*. If three or more vectors are parallel to the same plane, they are said to be *coplanar*.

Two vectors $\mathbf{P}$ and $\mathbf{Q}$ may be multiplied together in three different ways. The most general type of multiplication yields the undetermined product given by

$$\begin{aligned}\mathbf{PQ} = {} & P_xQ_x\mathbf{ii} + P_xQ_y\mathbf{ij} + P_xQ_z\mathbf{ik} \\ & + P_yQ_x\mathbf{ji} + P_yQ_y\mathbf{jj} + P_yQ_z\mathbf{jk} \\ & + P_zQ_x\mathbf{ki} + P_zQ_y\mathbf{kj} + P_zQ_z\mathbf{kk}.\end{aligned} \tag{3}$$

This product is neither vector nor scalar; it is known as a *dyad*.

The *vector* or *cross product* of two vectors is a vector perpendicular to their plane in the direction of advance of a right-handed screw when rotated from the first to the second of these vectors through the smaller angle between them. Its magnitude is equal to the product of the magnitudes of the two vectors by the sine of the angle between them. Therefore

$$\mathbf{P} \times \mathbf{Q} = -\mathbf{Q} \times \mathbf{P}. \tag{4}$$

Geometrically, this vector product has the magnitude of the parallelogram of which $\mathbf{P}$ and $\mathbf{Q}$ are the sides, and a direction at right angles to its surface. It follows from simple geometrical considerations that the distributive law holds for this product, that is,

$$(\mathbf{P} + \mathbf{Q}) \times \mathbf{R} = \mathbf{P} \times \mathbf{R} + \mathbf{Q} \times \mathbf{R}. \tag{5}$$

Therefore, inserting crosses between the vectors in each term of (3),

$$\mathbf{P} \times \mathbf{Q} = (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k}. \tag{6}$$

The *scalar* or *dot product* of two vectors is a scalar equal in magnitude to the product of the magnitudes of the two vectors by the cosine of the angle between them. Obviously the

distributive law holds for this product. Therefore, inserting dots between the vectors in each term of (3),

$$\mathbf{P}\cdot\mathbf{Q} = P_x Q_x + P_y Q_y + P_z Q_z. \tag{7}$$

The triple scalar product

$$(\mathbf{P} \times \mathbf{Q})\cdot\mathbf{R}$$

evidently measures the volume of the parallelepiped of which **P**, **Q**, and **R** are the edges. Hence the position of cross and dot in this product is immaterial, and its sign is changed by interchanging the positions of two adjacent vectors.

The triple vector product

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R}$$

is obviously a vector in the plane of **P** and **Q**. From simple geometrical considerations it follows that

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{P} = P^2\mathbf{Q} - \mathbf{P}\cdot\mathbf{Q}\mathbf{P}, \tag{8}$$

and

$$(\mathbf{P} \times \mathbf{Q}) \times \mathbf{Q} = \mathbf{P}\cdot\mathbf{Q}\mathbf{Q} - Q^2\mathbf{P}. \tag{9}$$

Now **R** may be written

$$\mathbf{R} = a\mathbf{P} + b\mathbf{Q} + c(\mathbf{P} \times \mathbf{Q}).$$

Therefore

$$\begin{aligned}(\mathbf{P} \times \mathbf{Q}) \times \mathbf{R} &= a(\mathbf{P} \times \mathbf{Q}) \times \mathbf{P} + b(\mathbf{P} \times \mathbf{Q}) \times \mathbf{Q} \\ &= (aP^2 + b\mathbf{P}\cdot\mathbf{Q})\,\mathbf{Q} - (a\mathbf{P}\cdot\mathbf{Q} + bQ^2)\,\mathbf{P} \\ &= \mathbf{P}\cdot\mathbf{R}\mathbf{Q} - \mathbf{Q}\cdot\mathbf{R}\mathbf{P}. \end{aligned} \tag{10}$$

This important expansion may be put in words as follows: Dot the exterior vector into the remoter vector inside the parentheses to form the scalar coefficient for the nearer one, then dot the exterior vector into the nearer vector to form the scalar coefficient for the remoter one, and subtract this result from the first.

The vector operator ∇ (read *del*) is of great importance in mathematical physics. This quantity is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Let ϕ be a scalar function of position in space. Then

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \tag{11}$$

is known as the *gradient* of ϕ. It may easily be shown to represent both in magnitude and direction the greatest (space) rate of increase of ϕ at the point in question.

Let $$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$

be a vector function of position in space. Then

$$\nabla\cdot\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{12}$$

is known as the *divergence* of **V**. If **V** is the flux of a fluid per unit time per unit cross section, the divergence of **V** is the excess of flux out of a unit volume over that into this volume. If the fluid is incompressible, the divergence is obviously zero except at those points where sources or sinks are present.

The vector

$$\nabla\times\mathbf{V} = \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \tag{13}$$

is known as the *curl* of **V**. If **V** specifies the linear velocities of the points of a rigid body, the curl is equal in magnitude and direction to twice the angular velocity of rotation.

The following identities may easily be verified by expansion:

$$\nabla\times\nabla\phi = 0, \tag{14}$$

$$\nabla\cdot\nabla\times\mathbf{V} = 0, \tag{15}$$

$// \nabla^2 V$

$$\nabla\times\nabla\times\mathbf{V} = \nabla\nabla\cdot\mathbf{V} - \nabla\cdot\nabla\mathbf{V}. \tag{16}$$

Gauss' Theorem. In treating vector integrals volume, surface, and line elements will be denoted respectively by $d\tau$, $d\sigma$, and $d\lambda$. The direction of an element of a closed surface will be taken as that of the outward-drawn normal, and the direction of an element of a closed curve will be taken as that in which a right-handed screw passing through the surface bounded by the curve must rotate in order to advance toward the positive side of this surface.

Let **V** be a vector function of position in space. Then Gauss' theorem states that

$$\int_\tau \nabla\cdot\mathbf{V}\,d\tau = \int_\sigma \mathbf{V}\cdot d\sigma, \tag{17}$$

where the surface integral is taken over the surface σ bounding the volume τ.

This theorem may be proved in the following way. In rectangular coördinates

$$\int \nabla\cdot\mathbf{V}\,d\tau = \int\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)dx\,dy\,dz.$$

Let x_1, y, z and x_2, y, z be the points of intersection of the surface bounding τ with a line parallel to the X axis. Then

$$\int_\tau \frac{\partial V_x}{\partial x}\,dx\,dy\,dz = \int\{V_x(x_2,\ y,\ z) - V_x(x_1,\ y,\ z)\}\,dy\,dz$$

$$= \int_\sigma V_x\,dy\,dz.$$

Therefore $$\int_\tau \nabla\cdot\mathbf{V}\,d\tau = \int(V_x\,dy\,dz + V_y\,dz\,dx + V_z\,dx\,dy)$$

$$= \int_\sigma \mathbf{V}\cdot d\sigma.$$

Stokes' Theorem. If **V** is a vector function of position in space, Stokes' theorem states that

$$\int_\sigma \nabla\times\mathbf{V}\cdot d\sigma = \int_\lambda \mathbf{V}\cdot d\lambda, \tag{18}$$

where the line integral is taken over the curve λ bounding the surface σ.

To prove this theorem proceed as follows. In rectangular coördinates

$$\int \nabla\times\mathbf{V}\cdot d\sigma = \int\left\{\left(\frac{\partial V_x}{\partial z}\,dz\,dx - \frac{\partial V_x}{\partial y}\,dx\,dy\right) + \left(\frac{\partial V_y}{\partial x}\,dx\,dy - \frac{\partial V_y}{\partial z}\,dy\,dz\right)\right.$$
$$\left. + \left(\frac{\partial V_z}{\partial y}\,dy\,dz - \frac{\partial V_z}{\partial x}\,dz\,dx\right)\right\}.$$

Let x, y_1, z_1 and x, y_2, z_2 be the points of intersection of the periphery of σ with a plane parallel to the YZ coördinate plane. Then, taking account of the signs of the differentials involved,

$$\int_\sigma\left(\frac{\partial V_x}{\partial z}dz\,dx-\frac{\partial V_x}{\partial y}dx\,dy\right)=\int_\sigma[\{V_x(x,\ y+dy,\ z+dz)$$
$$-V_x(x,\ y+dy,\ z)\}dx+\{V_x(x,\ y+dy,\ z)-V_x(x,\ y,\ z)\}dx]$$
$$=\int\{V_x(x,\ y_2,\ z_2)-V_x(x,\ y_1,\ z_1)\}dx$$
$$=\int_\lambda V_x dx.$$

Therefore $$\int_\sigma \nabla\times\mathbf{V}\cdot d\sigma=\int(V_x dx+V_y dy+V_z dz)$$
$$=\int_\lambda \mathbf{V}\cdot d\lambda.$$

Dyadics. A *dyadic* is a sum of a number of dyads. The first vector in each dyad is called the *antecedent*, and the second the *consequent*. Any dyadic may be reduced to the sum of three dyads. For if the dyadic ψ is given by

$$\psi=\mathbf{al}+\mathbf{bm}+\mathbf{cn}+\mathbf{do}, \tag{19}$$

the vector $\mathbf{o}$ may be written

$$\mathbf{o}=f\mathbf{l}+g\mathbf{m}+h\mathbf{n},$$

whence $$\psi=(\mathbf{a}+f\mathbf{d})\mathbf{l}+(\mathbf{b}+g\mathbf{d})\mathbf{m}+(\mathbf{c}+h\mathbf{d})\mathbf{n}. \tag{20}$$

Similarly, if either the antecedents or consequents of a dyadic are coplanar, the dyadic may be reduced to the sum of two dyads. Such a dyadic is said to be *planar*. If either antecedents or consequents are collinear, the dyadic becomes a single dyad and is said to be *linear*.

Consider the dyadic

$$\psi=\mathbf{al}+\mathbf{bm}+\mathbf{cn}.$$

If $\mathbf{P}$ is a vector,

$$\psi\cdot\mathbf{P}=\mathbf{al}\cdot\mathbf{P}+\mathbf{bm}\cdot\mathbf{P}+\mathbf{cn}\cdot\mathbf{P}$$

is also a vector. Dotting a dyadic into a vector, then, gives rise to a vector having a new direction and magnitude. This new

vector is a *linear vector function* of the original one. If a dyadic is planar, it will reduce to zero vectors having a certain direction, and if it is linear, it will cause all vectors parallel to a certain plane to vanish.

Obviously any dyadic may be written in the expanded form

$$\begin{aligned}\psi = {} & a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} \\ & + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} \\ & + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk}.\end{aligned} \tag{21}$$

It will now be shown that any dyadic may be put in such a form that its antecedents and consequents each constitute a right-handed set of mutually perpendicular vectors. Let $\boldsymbol{\alpha}$ be a unit vector of variable direction extending from the origin.

Then $$\boldsymbol{\beta} = \psi\cdot\boldsymbol{\alpha}$$

describes a closed surface about the origin as $\boldsymbol{\alpha}$ varies in direction. This surface may easily be shown to be an ellipsoid. Let **i** be the value of $\boldsymbol{\alpha}$ for which $\boldsymbol{\beta}$ assumes its maximum value **a**. Now consider all values of $\boldsymbol{\alpha}$ lying in the plane perpendicular to **i**. Let **j** be the value of $\boldsymbol{\alpha}$ in this plane for which $\boldsymbol{\beta}$ assumes its greatest value **b**. Finally, let **k** be a unit vector perpendicular to **i** and **j** in the sense that will make **i**, **j**, **k** a right-handed set. Let **c** be the value of $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$ equals **k**. Then, as the dyadic changes **i**, **j**, **k** into **a**, **b**, **c**, it may be written in the form

$$\psi = \mathbf{ai} + \mathbf{bj} + \mathbf{ck}.$$

Now $$\boldsymbol{\beta} = (\mathbf{ai} + \mathbf{bj} + \mathbf{ck})\cdot\boldsymbol{\alpha},$$

$$d\boldsymbol{\beta} = (\mathbf{ai} + \mathbf{bj} + \mathbf{ck})\cdot d\boldsymbol{\alpha},$$

$$\boldsymbol{\beta}\cdot d\boldsymbol{\beta} = \boldsymbol{\beta}\cdot\mathbf{ai}\cdot d\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot\mathbf{bj}\cdot d\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot\mathbf{ck}\cdot d\boldsymbol{\alpha}.$$

When $\boldsymbol{\alpha}$ is parallel to **i**, $\boldsymbol{\beta}$ has its maximum value **a**, and therefore

$$\mathbf{a}\cdot\mathbf{bj}\cdot d\boldsymbol{\alpha} + \mathbf{a}\cdot\mathbf{ck}\cdot d\boldsymbol{\alpha} = 0.$$

If, moreover, $d\boldsymbol{\alpha}$ is perpendicular to **j**, $\mathbf{a}\cdot\mathbf{c}$ vanishes, and if $d\boldsymbol{\alpha}$ is perpendicular to **k**, $\mathbf{a}\cdot\mathbf{b}$ vanishes. Hence **a** is perpendicular to both **b** and **c**.

Now let $\boldsymbol{\alpha}$ be restricted to the $\mathbf{jk}$ plane. Then

$$\boldsymbol{\beta}\cdot d\boldsymbol{\beta} = \boldsymbol{\beta}\cdot\mathbf{bj}\cdot d\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot\mathbf{ck}\cdot d\boldsymbol{\alpha}.$$

When $\boldsymbol{\alpha}$ is parallel to $\mathbf{j}$, $\boldsymbol{\beta}$ has its greatest value $\mathbf{b}$, and therefore $\mathbf{b}\cdot\mathbf{c}$ vanishes. Therefore $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually perpendicular. If they do not form a right-handed set, the direction of one of them may be reversed provided its sign is changed. Hence, if $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ constitute a right-handed set of mutually perpendicular unit vectors parallel respectively to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the dyadic may be written

$$\boldsymbol{\psi} = a\mathbf{i}_1\mathbf{i} + b\mathbf{j}_1\mathbf{j} + c\mathbf{k}_1\mathbf{k}. \tag{22}$$

If

$$\boldsymbol{\psi} = \mathbf{al} + \mathbf{bm} + \mathbf{cn},$$

the *conjugate* of $\boldsymbol{\psi}$ is defined as

$$\boldsymbol{\psi}_c = \mathbf{la} + \mathbf{mb} + \mathbf{nc},$$

and a dyadic is said to be *self-conjugate* if it is equal to its conjugate. Obviously, if $\boldsymbol{\psi}$ is self-conjugate,

$$\boldsymbol{\psi}\cdot\mathbf{P} = \mathbf{P}\cdot\boldsymbol{\psi}. \tag{23}$$

The *idemfactor* $\mathbf{I}$ is defined as

$$\begin{aligned}\mathbf{I} &= \mathbf{ii} + \mathbf{jj} + \mathbf{kk}\\ &= \mathbf{i}_1\mathbf{i}_1 + \mathbf{j}_1\mathbf{j}_1 + \mathbf{k}_1\mathbf{k}_1.\end{aligned}$$

Evidently this dyadic is self-conjugate, and moreover

$$\mathbf{I}\cdot\mathbf{P} = \mathbf{P}. \tag{24}$$

It will now be shown that any complete self-conjugate dyadic may be written in the form

$$\boldsymbol{\psi} = a\mathbf{ii} + b\mathbf{jj} + c\mathbf{kk}, \tag{25}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ constitute a right-handed set of mutually perpendicular unit vectors.

It has already been shown that the dyadic may be written

$$\boldsymbol{\psi} = a\mathbf{i}_1\mathbf{i} + b\mathbf{j}_1\mathbf{j} + c\mathbf{k}_1\mathbf{k}.$$

Now, as $\boldsymbol{\psi}$ is self-conjugate,

$$\begin{aligned}\boldsymbol{\psi}_c &= \boldsymbol{\psi}\\ &= a\mathbf{ii}_1 + b\mathbf{jj}_1 + c\mathbf{kk}_1.\end{aligned}$$

Form the products

$$\psi\cdot\psi_c = a^2 \mathbf{i}_1\mathbf{i}_1 + b^2 \mathbf{j}_1\mathbf{j}_1 + c^2 \mathbf{k}_1\mathbf{k}_1,$$

and

$$\psi_c\cdot\psi = a^2 \mathbf{i}\mathbf{i} + b^2 \mathbf{j}\mathbf{j} + c^2 \mathbf{k}\mathbf{k}.$$

These are equal, as the dyadic is self-conjugate. Therefore put

$$\psi^2 = \psi\cdot\psi_c = \psi_c\cdot\psi.$$

Now consider the dyadic

$$\psi^2 - a^2\mathbf{I} = (b^2 - a^2)\,\mathbf{j}_1\mathbf{j}_1 + (c^2 - a^2)\,\mathbf{k}_1\mathbf{k}_1.$$

Obviously $$(\psi^2 - a^2\mathbf{I})\cdot\mathbf{i}_1 = 0.$$

But, as $$\psi^2 - a^2\mathbf{I} = (b^2 - a^2)\,\mathbf{j}\mathbf{j} + (c^2 - a^2)\,\mathbf{k}\mathbf{k},$$

it follows that $$(\psi^2 - a^2\mathbf{I})\cdot\mathbf{i} = 0.$$

Hence, as $(\psi^2 - a^2\mathbf{I})$ is planar, but not linear, $\mathbf{i}_1$ must be parallel to $\mathbf{i}$. Similarly, $\mathbf{j}_1$ must be parallel to $\mathbf{j}$, and $\mathbf{k}_1$ to $\mathbf{k}$.

CHAPTER I

THE PRINCIPLE OF RELATIVITY

1. Motion. The concept of motion comprises two essential factors: a *moving element*, and a *reference body* relative to which the motion takes place. A grain of sand lying on the floor of a railway car is not in motion at all if the car itself is chosen as reference body, although it may be moving rapidly relative to the earth. If, however, the car, the earth, and all other objects save the grain of sand are removed, the lack of a reference body makes it impossible to form a conception of motion.

A moving element is characterized by a point—whether in a material body or not—which can be continuously identified. In the following discussion a point always will be understood to have this property. A reference body is essentially a group of points along the path of a moving element, together with a device for assigning numerical values to the intervals of time between coincidences of the moving element with successive points of the body, and to the distances between these points. For such characteristics are obviously necessary in order to make possible quantitative evaluation of the motion of the moving element.

2. Reference system. A *reference system* is an assemblage of points filling all space. A device is provided for indicating time at these points in such a way as to assure synchronism according to some arbitrary standard, and for measuring distances between them. This device is subject to the following conditions, but otherwise it is quite arbitrary:

(1) Two points which are in synchronism with a third are also in synchronism with each other.

(2) The distance between two points is independent of the time at which it is measured.

Thus a reference system serves as a reference body for any moving element. It must not, however, be imagined to offer any obstruction to the motion through it of such an element, or of another reference system.

A material body of finite extent may be considered to constitute a reference system if the points of the body itself are supposed to have points outside associated with them in such a way that the whole assemblage possesses the properties described above. In order that the material part of such a system shall in no degree obstruct the motion through it of a moving element, those portions of it which would be in the way may be regarded as temporarily removed.

The motion of a given moving element may be described relative to an infinite number of reference systems. However, these systems are not in general of the same significance. For let A, B, and C be three systems from which the motion of the moving element P may be observed. Suppose it is found that the motion of P relative to A is conditioned by that of B, but is independent of that of C. In such a case the motion of P is said to be *related* to B, which is known as a *related reference system.* C, on the other hand, is an *unrelated* or *ideal reference system.* Thus for the motion of a shot, the gun from which it is fired constitutes a related reference system. The velocity of a sound wave is determined, not by the motion of the source, but by the characteristics of the medium through which it passes. Hence in this case the source is an ideal reference body, while the medium is a related one.

3. Principle of relativity. In the case of light, it has been generally recognized, ever since the vindication of the wave theory by Young and Fresnel, that the source does not constitute a related reference system. Recent analysis of the observed motion of certain double stars has confirmed this supposition. But most physicists have felt it necessary to postulate the existence of an all-pervading medium in order to form a mental picture of the propagation of light waves through otherwise empty space. For a long time they were accustomed to attribute

to this medium, known as the *ether*, the properties of a related reference system. Finally Michelson devised an experimental method of measuring the velocity of the earth relative to the ether, based on the assumption that the ether is a related reference system for the motion of light. Much to everyone's surprise, this velocity turned out to be zero. Excluding the possibility of the earth's being at rest relative to the ether, and one or two other equally improbable explanations, the only conclusion to be reached was that the assumption that the ether is a related reference system for the motion of light was unjustified. The inference to be drawn from the result of this experiment, then, may be embodied in the following "principle of negation."

For the motion of an effect which travels through empty space, such as a light wave or one of the moving elements which form an electromagnetic or a gravitational field, there is no related reference system.

An immediate consequence is contained in the following statement.

If a law governing physical phenomena which are conditioned solely by those effects which travel through empty space, is determined from observations made in two different reference systems, the form of this law and the values of the constants entering into it can differ in the two cases only in so far as the geometry and devices for measuring time and distance, together with the units of these quantities, may differ in the two systems. Their relative motion in itself can affect neither the form of the law nor the values of the constants involved. This is the principle of general relativity.

Consider two reference systems which have the same geometry, devices of the same character for measuring time and distance, and interchangeable units of these quantities. Such systems may be said to be *reciprocal.* It follows that

A law governing physical phenomena which are conditioned solely by those effects which travel through empty space, has the same form and its constants have the same values for two mutually

reciprocal systems. In the subsequent discussion the phrase "principle of relativity" will be understood to refer to this restricted form of the general principle.

4. Reciprocal systems. Consider two reciprocal Euclidean systems S and S', such that all points of S' have the same constant velocity $\mathbf{v}$ relative to S. Let light travel in straight lines in S with a constant speed c. Then the principle of relativity requires that light shall travel in straight lines in S' with the same constant speed c. Let A and B be two points of either system a distance Δr apart. Since the speed of light is the same in all directions, the time Δt taken by a light wave in passing from A to B, as measured in the system in which these two points are located, is the same as that taken by a light wave in travelling from B to A. Moreover,

$$\Delta r = c\Delta t. \tag{1}$$

5. Differential transformations. Let a set of right-handed axes XYZ be fixed in S so that the X axis has the direction of the velocity of S'. Let a similar set of axes $X'Y'Z'$, parallel to XYZ respectively, be fixed in S'. Let x, y, z, and t be the coördinates of a point and the time at the point as measured in S, and x', y', z', and t' the corresponding quantities as measured in S'. It is desired to obtain x', y', z', and t' as functions of x, y, z, and t. Let A' and B' be two neighboring points of S'. A light wave leaving A' at the time t arrives at B' at the time $t + dt_1$, and one leaving B' at the same time t reaches A' at the time $t + dt_2$, these times being measured in S. If the coördinates of A' and B' relative to S at the time t are denoted by x, y, z and $x + dx$, $y + dy$, $z + dz$ respectively, the time dt' taken by the first wave to travel from A' to B' as measured in S' is

$$dt' = \frac{\partial t'}{\partial x}(dx + v dt_1) + \frac{\partial t'}{\partial y} dy + \frac{\partial t'}{\partial z} dz + \frac{\partial t'}{\partial t} dt_1,$$

and the equal time taken by the second wave in passing from B' to A' is

$$dt' = \frac{\partial t'}{\partial x}(- dx + v dt_2) - \frac{\partial t'}{\partial y} dy - \frac{\partial t'}{\partial z} dz + \frac{\partial t'}{\partial t} dt_2.$$

Subtracting,

$$\frac{\partial t'}{\partial x}\left\{dx+\frac{1}{2}v(dt_1-dt_2)\right\}+\frac{\partial t'}{\partial y}dy+\frac{\partial t'}{\partial z}dz+\frac{1}{2}\frac{\partial t'}{\partial t}(dt_1-dt_2)=0, \quad (2)$$

and adding,

$$dr'=cdt'=\frac{1}{2}c\left\{v\frac{\partial t'}{\partial x}+\frac{\partial t'}{\partial t}\right\}(dt_1+dt_2). \quad (3)$$

But the squares of the distances travelled by the wave going from A' to B' and by that passing from B' to A' as measured in S are respectively

$$c^2dt_1^2=(dx+vdt_1)^2+dy^2+dz^2,$$

$$c^2dt_2^2=(dx-vdt_2)^2+dy^2+dz^2.$$

Hence

$$dt_1-dt_2=\frac{2v}{c^2-v^2}dx, \quad (4)$$

$$dt_1+dt_2=\frac{2\sqrt{c^2-v_\perp^2}}{c^2-v^2}dr, \quad (5)$$

where $dr^2\equiv dx^2+dy^2+dz^2$, and $v_\perp$ is the component of v at right angles to dr. Substituting this value of dt_1-dt_2 in (3),

$$\frac{1}{1-\beta^2}\left(\frac{\partial t'}{\partial x}+\frac{\beta}{c}\frac{\partial t'}{\partial t}\right)dx+\frac{\partial t'}{\partial y}dy+\frac{\partial t'}{\partial z}dz=0,$$

where $\beta\equiv\dfrac{v}{c}\cdot$

Now dx, dy, and dz are arbitrary. Hence their coefficients must vanish, that is,

$$\frac{\partial t'}{\partial x}=-\frac{\beta}{c}\frac{\partial t'}{\partial t}, \quad (6)$$

$$\frac{\partial t'}{\partial y}=0, \quad (7)$$

$$\frac{\partial t'}{\partial z}=0. \quad (8)$$

Therefore the complete differential dt' is given by

$$dt'=\frac{\partial t'}{\partial t}dt+\frac{\partial t'}{\partial x}dx+\frac{\partial t'}{\partial y}dy+\frac{\partial t'}{\partial z}dz$$

$$=\frac{\partial t'}{\partial t}\left(dt-\frac{\beta}{c}dx\right). \quad (9)$$

To obtain dr', substitute (5) and (6) in (3). Then

$$dr' = \frac{\partial t'}{\partial t}\sqrt{1 - \frac{v_1^2}{c^2}}\, dr.$$

Giving dr the values dx_1, dy, and dz, it is seen that

$$dx' = \frac{\partial t'}{\partial t}\, dx_1, \tag{10}$$

$$dy' = \frac{\partial t'}{\partial t}\sqrt{1 - \beta^2}\, dy, \tag{11}$$

$$dz' = \frac{\partial t'}{\partial t}\sqrt{1 - \beta^2}\, dz. \tag{12}$$

Now dx_1 in (10) is the distance of B' from A' when the time is the same at the two points. If dx is the distance of the position of B' at the time $t + dt$ from that of A' at the time t,

$$dx_1 = dx - vdt.$$

So (10) becomes

$$dx' = \frac{\partial t'}{\partial t}(dx - vdt).$$

Since the units of length in S and S' are interchangeable, it follows from symmetry that $dy' = dy$, and $dz' = dz$. Hence if

$$k \equiv \frac{1}{\sqrt{1 - \beta^2}},$$

$$\frac{\partial t'}{\partial t} = k.$$

Hence the differential transformations between the two systems are

$$dt' = k\left(dt - \frac{\beta}{c}\, dx\right), \quad \text{or} \quad dt = k\left(dt' + \frac{\beta}{c}\, dx'\right), \tag{13}$$

$$dx' = k(dx - vdt), \qquad dx = k(dx' + vdt'), \tag{14}$$

$$dy' = dy, \qquad dy = dy', \tag{15}$$

$$dz' = dz, \qquad dz = dz', \tag{16}$$

where the second column is obtained from the first by solving for the unprimed differentials, or by changing the sign of v. From these expressions it follows that

$$dr'^2 - c^2dt'^2 = dr^2 - c^2dt^2$$

is an invariant of the transformation.

Now the velocity v' of a point of S relative to S' is obtained by dividing dx' by dt', x remaining constant. Therefore

$$v' = -v,$$

as might have been expected from considerations of symmetry.

6. Space and time transformations. Integrating the differential relations (13) to (16), and determining the constants of integration on the assumption that the origins of the two systems are in coincidence when the time at each is zero,

$$t' = k\left(t - \frac{\beta}{c}x\right), \quad \text{or} \quad t = k\left(t' + \frac{\beta}{c}x'\right), \tag{17}$$

$$x' = k(x - vt), \qquad x = k(x' + vt'), \tag{18}$$

$$y' = y, \qquad y = y', \tag{19}$$

$$z' = z, \qquad z = z'. \tag{20}$$

These four relations are known as the Lorentz-Einstein transformations.

Consider a moving element whose velocity components relative to S are V_x, V_y, and V_z. From the differential relations of the preceding section it follows at once that

$$V'_x = \frac{V_x - v}{1 - \beta\dfrac{V_x}{c}}, \quad \text{or} \quad V_x = \frac{V'_x + v}{1 + \beta\dfrac{V'_x}{c}}, \tag{21}$$

$$V'_y = \frac{V_y}{k\left(1 - \beta\dfrac{V_x}{c}\right)}, \qquad V_y = \frac{V'_y}{k\left(1 + \beta\dfrac{V'_x}{c}\right)}, \tag{22}$$

$$V'_z = \frac{V_z}{k\left(1 - \beta\dfrac{V_x}{c}\right)}, \qquad V_z = \frac{V'_z}{k\left(1 + \beta\dfrac{V'_x}{c}\right)}. \tag{23}$$

For the resultant velocity

$$V'^2 = c^2\left\{1 - \frac{1-\dfrac{V^2}{c^2}}{k^2\left(1-\beta\dfrac{V_x}{c}\right)^2}\right\}.$$

Hence if $V = c$, $V' = c$ also, as should be. Now

$$V'^2 - V^2 = c^2\left(1-\frac{V^2}{c^2}\right)\left\{1 - \frac{1-\beta^2}{\left(1-\beta\dfrac{V_x}{c}\right)^2}\right\},$$

so the velocity of light is the only velocity independent of v which is the same for the two systems.

Suppose that the velocity of S' relative to S is nearly as great as the velocity of light. Then $\beta = 1 - \delta$, where δ is small. Consider a body moving in the X' direction with a velocity relative to S' only slightly less than the velocity of light. Then $V'_x = c(1-\epsilon)$, where ϵ is small. Equation (21) gives for the velocity of this body relative to S,

$$V_x = c\frac{2-\delta-\epsilon}{2-\delta-\epsilon+\delta\epsilon}$$

$$\doteqdot c\left(1-\frac{\delta\epsilon}{2}\right),$$

whence V_x is less than c. For example, if $v = 0.9\,c$, and $V'_x = 0.9\,c$, V_x would be $1.8\,c$ according to nineteenth-century conceptions of space and time. But the addition theorem of velocities just obtained from the principle of relativity gives $V_x = 0.994\,c$.

The relations between components of acceleration as measured on the two systems are obtained by differentiating equations (21) to (23) with respect to the time, remembering that

$$\frac{d}{dt'} = \frac{dt}{dt'}\frac{d}{dt}$$

$$= \frac{1}{k\left(1-\beta\dfrac{V_x}{c}\right)}\frac{d}{dt}.$$

These relations are

$$f'_x = \frac{f_x}{k^3\left(1-\beta\frac{V_x}{c}\right)^3}, \qquad \text{or} \qquad f_x = \frac{f'_x}{k^3\left(1+\beta\frac{V'_x}{c}\right)^3}, \tag{24}$$

$$f'_y = \frac{f_y - \frac{\beta}{c}(f_y V_x - f_x V_y)}{k^2\left(1-\beta\frac{V_x}{c}\right)^3}, \qquad f_y = \frac{f'_y + \frac{\beta}{c}(f'_y V'_x - f'_x V'_y)}{k^2\left(1+\beta\frac{V'_x}{c}\right)^3}, \tag{25}$$

$$f'_z = \frac{f_z - \frac{\beta}{c}(f_z V_x - f_x V_z)}{k^2\left(1-\beta\frac{V_x}{c}\right)^3}, \qquad f_z = \frac{f'_z + \frac{\beta}{c}(f'_z V'_x - f'_x V'_z)}{k^2\left(1+\beta\frac{V'_x}{c}\right)^3}. \tag{26}$$

Suppose that the moving element is at rest in S'. Then the second column becomes

$$f_x = \frac{f'_x}{k^3}, \tag{27}$$

$$f_y = \frac{f'_y}{k^2}, \tag{28}$$

$$f_z = \frac{f'_z}{k^2}. \tag{29}$$

7. Four-dimensional representation. The Lorentz-Einstein transformations can be represented very simply by a rotation in a four-dimensional manifold. For consider a set of rectangular axes $XYZL$ in four-dimensional space, such that the distances x, y, z of a moving element from the origin of S are measured along the first three axes, and the quantity $l \equiv ict$ along the fourth axis, where $i \equiv \sqrt{-1}$. The position of a moving element at a given time is represented by a point in this space, and the locus of the positions of such an element at successive instants by a line. This line is called the *world line* of the moving element. Thus the world line of a body permanently at rest relative to S is a straight line parallel to the L axis. The world

line of a point moving with velocity V relative to S is inclined to the L axis by an angle ϕ such that

$$\tan\phi = -i\frac{V}{c}.$$

Hence the world line of a point of S' is parallel to the XL plane and makes with the L axis an angle α given by

$$\tan\alpha = -i\beta, \quad \sin\alpha = -ik\beta, \quad \cos\alpha = k.$$

Therefore in terms of x, y, z, and l the Lorentz-Einstein transformations take the form

$$l' = l\cos\alpha + x\sin\alpha, \quad \text{or} \quad l = l'\cos\alpha - x'\sin\alpha, \tag{30}$$

$$x' = x\cos\alpha - l\sin\alpha, \qquad x = x'\cos\alpha + l'\sin\alpha, \tag{31}$$

$$y' = y, \qquad y = y', \tag{32}$$

$$z' = z, \qquad z = z', \tag{33}$$

amounting formally to a rotation of the X and L axes through the imaginary angle α.

CHAPTER II

THE RETARDED FIELD OF A POINT CHARGE

8. Electric field. Continuous lines may be imagined to spread out from every elementary electric charge in such a way as to diverge uniformly in all directions when viewed from the system in which, at the instant considered, the charge is at rest. These lines are called *lines of force* and, taken together, they constitute the charge's *field*. The number of lines emanating from an element of charge de will be supposed to be very large, no matter how small de may be. A bundle of M lines, where M is a very large number, will be considered to constitute a *tube of force*. The *field strength*, or *electric intensity*, **E**, at a point in a field, is a vector having the direction of the lines of force at that point and equal in magnitude to the number of tubes per unit cross section. Thus if dN tubes pass through a small surface of area $d\sigma$ whose normal makes an angle θ (Fig. 1) with the field, the magnitude of the field strength is given by

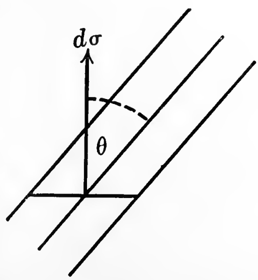

FIG. 1

$$E \equiv \frac{dN}{d\sigma \cos\theta}.$$

Hence the component of the field strength parallel to $d\boldsymbol{\sigma}$ is

$$E_{d\sigma} \equiv \frac{dN}{d\sigma},$$

or, in vector notation, $\mathbf{E}_{d\sigma} \equiv \dfrac{dN}{d\sigma^2} d\boldsymbol{\sigma}.$

The ratio of the number of tubes of force diverging from a charge to the magnitude of the charge determines the unit of

charge. The simplest unit, and the one which will be used in the following pages, is that which makes the charge at any point equal to the number of tubes of force diverging from that point. This unit is the one advocated by Heaviside and Lorentz and, as will appear later, is smaller than the usual electrostatic unit by the factor

$$\frac{1}{\sqrt{4\pi}}.$$

9. Motion of a field. Consider an electric field which is being observed from the two reference systems S and S' of the previous chapter. The principle of relativity requires that the velocity of the moving elements comprising the field shall have the same numerical value no matter whether observations are carried on in S, S', or some other system reciprocal to one of these. In section 6 it was shown that the velocity of light is the only velocity which satisfies this condition. Hence the moving elements constituting an electric field must have the velocity of light.

Suppose a charged particle to be permanently at rest in S. Although the moving elements constituting its field are in motion with the velocity of light, the lines of force themselves are stationary. Hence the motion must be entirely along these lines. Now consider a charged particle moving with a constant velocity $\mathbf{V}$ relative to S. As the charge carries its field along with it, the velocity of a moving element will be along the lines of force only at points in the line of motion. At all other points this velocity will make an angle with the lines of force which will be greater the greater the speed. Therefore in general the complete specification of an electric field due to a charged particle requires the knowledge at every point of the values of two vectors, the field strength $\mathbf{E}$ and the velocity $\mathbf{c}$. Both magnitude and direction of $\mathbf{E}$ must be given, but as the magnitude of $\mathbf{c}$ is known its direction only is required.

10. Transformation equations. Suppose that the field strength $\mathbf{E}$ and velocity $\mathbf{c}$ relative to S are known for the field due to a point charge. It is desired to find the values of these quantities

as measured in S'. Let P and Q (Fig. 2) be two neighboring points on the same line of force, the coördinates of Q relative to P being dx, dy, dz when the time is the same at the two points according to the standards of S. Then if the time at Q should be later by dt than that at P, the coördinates of Q relative to P would become

$$dx_1 = dx + c_x dt,$$
$$dy_1 = dy + c_y dt,$$
$$dz_1 = dz + c_z dt.$$

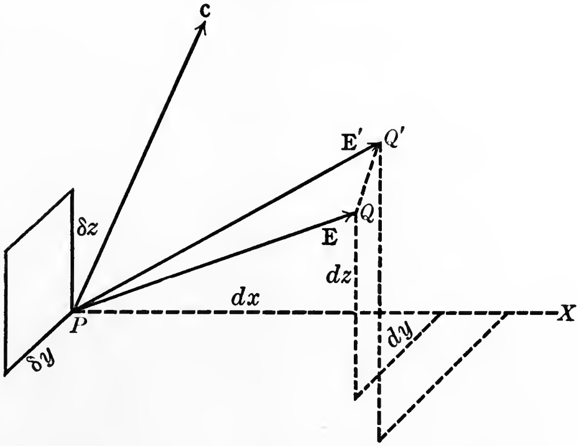

FIG. 2

Now, in order that the times at P and Q should be the same when measured in S', equation (13) of section 5 shows it to be necessary that

$$dt = \frac{\beta}{c} dx_1.$$

Hence when $dt' = 0$,

$$dx_1 = \frac{dx}{1 - \beta\frac{c_x}{c}},$$

$$dy_1 = dy + \frac{\beta\frac{c_y}{c}dx}{1 - \beta\frac{c_x}{c}},$$

$$dz_1 = dz + \frac{\beta\frac{c_z}{c}dx}{1 - \beta\frac{c_x}{c}}.$$

Substituting in the transformation equations (14) to (16), second column, of section 5,

$$dx' = \frac{dx}{k\left(1-\beta\frac{c_x}{c}\right)},$$

$$dy' = dy + \frac{\beta\frac{c_y}{c}dx}{1-\beta\frac{c_x}{c}},$$

$$dz' = dz + \frac{\beta\frac{c_z}{c}dx}{1-\beta\frac{c_x}{c}},$$

are found to be the coördinates of Q relative to P as measured in S' when the time is the same at the two points according to the standards of this system. The position of Q at this time is shown by Q' on the figure. Consequently the lines of force in S' extend from P to Q', instead of from P to Q as in S.

Let δN be the number of tubes of force passing through the area $\delta y \delta z$ in S. Then, since $\delta y' = \delta y$, and $\delta z' = \delta z$,

$$E'_x = \frac{\delta N}{\delta y'\delta z'} = \frac{\delta N}{\delta y \delta z} = E_x .$$

Moreover,
$$\frac{E_x}{dx} = \frac{E_y}{dy} = \frac{E_z}{dz},$$

$$\frac{E'_x}{dx'} = \frac{E'_y}{dy'} = \frac{E'_z}{dz'} .$$

Hence
$$E'_y = kE_x \frac{dy - \beta\left(\frac{c_x}{c}dy - \frac{c_y}{c}dx\right)}{dx}$$

$$= k\left\{E_y - \beta\left(\frac{c_x}{c}E_y - \frac{c_y}{c}E_x\right)\right\}$$

$$= k\left\{E_y - \frac{\beta}{c}(\mathbf{c}\times\mathbf{E})_z\right\},$$

and similarly,
$$E'_z = k\left\{E_z + \frac{\beta}{c}(\mathbf{c}\times\mathbf{E})_y\right\}.$$

The *magnetic intensity* **H** is defined by

$$\mathbf{H} \equiv \frac{1}{c}(\mathbf{c} \times \mathbf{E}).$$

Therefore
$$E'_x = E_x, \tag{1}$$
$$E'_y = k\{E_y - \beta H_z\} = k\left\{E_y + \frac{1}{c}(\mathbf{v} \times \mathbf{H})_y\right\}, \tag{2}$$
$$E'_z = k\{E_z + \beta H_y\} = k\left\{E_z + \frac{1}{c}(\mathbf{v} \times \mathbf{H})_z\right\}, \tag{3}$$

where the primed and unprimed quantities may be interchanged, provided the sign of v is changed.

The transformations for **c** are obtained at once from the equations (21), (22), and (23) of section 6 for transforming velocity components. They are

$$c'_x = \frac{c_x - v}{1 - \beta\frac{c_x}{c}}, \tag{4}$$

$$c'_y = \frac{c_y}{k\left(1 - \beta\frac{c_x}{c}\right)}, \tag{5}$$

$$c'_z = \frac{c_z}{k\left(1 - \beta\frac{c_x}{c}\right)}. \tag{6}$$

From the transformations for the components of **E** and **c** those for the components of **H** are readily deduced. For

$$H'_x \equiv \frac{c'_y}{c}E'_z - \frac{c'_z}{c}E'_y$$

by definition. Substituting the values of the components of **E′** and **c′** in terms of the unprimed quantities in this identity and in the corresponding expressions for H_y and H_z, it is found that

$$H'_x = H_x, \tag{7}$$
$$H'_y = k\{H_y + \beta E_z\} = k\left\{H_y - \frac{1}{c}(\mathbf{v} \times \mathbf{E})_y\right\}, \tag{8}$$
$$H'_z = k\{H_z - \beta E_y\} = k\left\{H_z - \frac{1}{c}(\mathbf{v} \times \mathbf{E})_z\right\}. \tag{9}$$

A field formed by the superposition of the individual fields of a number of charged particles is termed a *complex* field, in

contradistinction to the *simple field* of a single elementary charge, and the electric and magnetic intensities in such a field are defined as the vector sums of the corresponding intensities of the component simple fields. It is to be noted, however, that the velocity $\mathbf{c}$, in so far as its direction is concerned, always refers to the field of an element of charge, and never to the resultant of a number of such simple fields superposed. If it is desired to avoid explicit reference to the components of a complex field, the field must be described by means of equations which do not involve the direction of motion of the constituent moving elements.

Since the transformations that have been obtained for $\mathbf{E}$ and $\mathbf{H}$ due to a single point charge are linear in these quantities, they apply as well to complex as to simple fields.

11. Point charge at rest. Let e (Fig. 3) be a point charge momentarily at rest at the origin of S at the time 0. It is desired to find the field strength $\mathbf{E}$ at P at a time $t=r/c$, where r is the distance of P from O. Two moving elements leaving e in slightly different directions at the time 0 will be at P and Q at the time t.

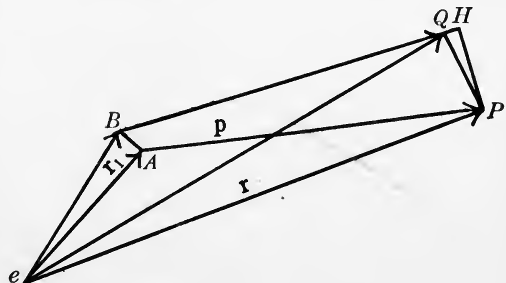

Fig. 3

At a time dt, e will still be at O (to the second order of small quantities). Consequently a moving element coming from e at this time and belonging to the same line of force as that at P will reach some such point as A by the time t. Similarly, one belonging to the same line of force as that at Q will reach B by the time t. Hence, if r_1 denotes the distance $\overline{OA}$,

$$\mathbf{r}=\mathbf{c}t,$$

and

$$\mathbf{r}_1=(\mathbf{c}+d\mathbf{c})(t-dt)$$
$$=\mathbf{c}t-\mathbf{c}\,dt+t\,d\mathbf{c},$$

where $|\mathbf{c}+d\mathbf{c}|=|\mathbf{c}|$ necessarily, or $d\mathbf{c}$ is perpendicular to $\mathbf{c}$.

Put $$\mathbf{p} \equiv \mathbf{r} - \mathbf{r}_1.$$

Then $$\mathbf{p} = \left(\mathbf{c}dt - \frac{r}{c}d\mathbf{c}\right),$$

$$\mathbf{p}\cdot\mathbf{r} = crdt.$$

Now since the lines of force, as viewed from S, diverge uniformly in all directions from O at the time 0, the number of tubes per unit area passing through a small surface at P with normal parallel to $\mathbf{r}$ is

$$\frac{e}{4\pi r^2}.$$

Hence $$\mathbf{E} = \frac{e}{4\pi r^2}\frac{\mathbf{p}}{p\cos QPH}$$

$$= \frac{e}{4\pi r}\frac{\mathbf{p}}{\mathbf{p}\cdot\mathbf{r}}$$

$$= \frac{e}{4\pi r^2 c}\left\{\mathbf{c} - \frac{r}{c}\frac{d\mathbf{c}}{dt}\right\}. \qquad (10)$$

Suppose that the charge e has an acceleration $\mathbf{f}$ relative to S. Then at the time dt it will be at rest in some reciprocal system S', which has a velocity $\mathbf{f}dt$ relative to S. As the lines of force diverge uniformly from the charge when viewed from the system in which, for the instant, the charge is at rest, and as the velocities of the moving elements constituting the portions of these lines in the immediate vicinity of the charge are along the lines themselves when observed from this system, it follows that if two moving elements, one of which leaves e at the time 0 and the other at the time dt, are to lie on the same line of force, the velocity of the second must make the same angle in S' with the direction of $\mathbf{f}$ as that of the first does in S. If the velocities of these two elements are denoted by $\mathbf{c}'$ and $\mathbf{c}$, and if the X' and X axes are taken parallel to $\mathbf{f}$,

$$c'_x = c_x.$$

But, from (4), $$c'_x = \frac{(c_x + dc_x) - fdt}{1 - \dfrac{c_x f dt}{c^2}}.$$

Let α (Fig. 4) be the angle between **c** and **f**. Then

$$c_x = c \cos \alpha,$$

$$dc_x = -c \sin \alpha \, d\alpha.$$

Hence
$$\cos \alpha = \frac{\cos \alpha - \sin \alpha \, d\alpha - \dfrac{fdt}{c}}{1 - \dfrac{fdt}{c} \cos \alpha},$$

or
$$d\alpha = -\frac{fdt}{c} \sin \alpha.$$

Now
$$dc = c\,d\alpha$$
$$= -fdt \sin \alpha,$$

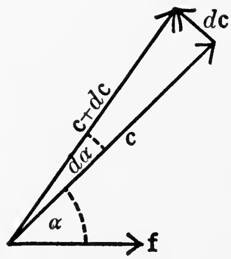

FIG. 4

which becomes, in vector notation,

$$d\mathbf{c} = -\frac{(\mathbf{f} \times \mathbf{c}) \times \mathbf{c}}{c^3} dt. \tag{11}$$

Substituting in (10) it is seen that if a charge e which has an acceleration **f** is momentarily at rest at O at a time 0, the field strength at a point P distant r from O at a time r/c is given by

$$\mathbf{E} = \left[\frac{e}{4\pi r^2 c} \left[\mathbf{c} + \frac{r}{c^3} (\mathbf{f} \times \mathbf{c}) \times \mathbf{c} \right] \right], \tag{12}$$

where the heavy brackets are used to denote the fact that the quantities contained therein are *retarded*; that is, these quantities refer to the *effective* position of the charge, or its position at a time r/c earlier than that for which the field strength at P is to be determined.

Since
$$\mathbf{H} \equiv \frac{1}{c} (\mathbf{c} \times \mathbf{E})$$

by definition,
$$\mathbf{H} = \left[\frac{e}{4\pi r^2 c} \left[-\frac{r}{c^4} \{ (\mathbf{f} \times \mathbf{c}) \times \mathbf{c} \} \times \mathbf{c} \right] \right]. \tag{13}$$

12. Point charge in motion. Consider a point charge e which is passing the origin of S at the time 0 with velocity **v** and acceleration **f**. Choose axes so that **v** is along the X axis. Then

this charge is at rest, at the instant considered, in a system S' which has a velocity v in the X direction relative to S. So

$$\mathbf{E}' = \frac{e}{4\pi r'^2 c}\left[\mathbf{c}' + \frac{r'}{c^3}(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'\right]$$

and

$$\mathbf{H}' = \frac{e}{4\pi r'^2 c}\left[-\frac{r'}{c^4}\{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'\} \times \mathbf{c}'\right]$$

are the values of $\mathbf{E}'$ and $\mathbf{H}'$ at a point P distant r' from O at the time r'/c. It is desired to determine $\mathbf{E}$ and $\mathbf{H}$ at P at this same instant. Since the velocity of light is the same in the two systems, the time at P will be r/c in S when it is r'/c in S'. Hence the result of the transformation about to be carried through will give $\mathbf{E}$ and $\mathbf{H}$ at P at the time r/c.

Let α' be the angle in S' which the line $\overline{OP}$ makes with the X' axis. Without loss of generality this line may be supposed to lie in the $X'Y'$ plane. The Lorentz-Einstein transformations give

$$r' \cos \alpha' = kr(\cos \alpha - \beta),$$

$$r' \sin \alpha' = r \sin \alpha;$$

whence

$$r' = kr(1 - \beta \cos \alpha),$$

$$\cos \alpha' = \frac{\cos \alpha - \beta}{1 - \beta \cos \alpha},$$

$$\sin \alpha' = \frac{\sin \alpha}{k(1 - \beta \cos \alpha)}.$$

From (4), (5), and (6), $c'_x = \dfrac{c_x - v}{1 - \beta \cos \alpha}$,

$$c'_y = \frac{c_y}{k(1 - \beta \cos \alpha)},$$

$$c'_z = 0;$$

and from (27), (28), and (29) of section 6,

$$f'_x = k^3 f_x,$$

$$f'_y = k^2 f_y,$$

$$f'_z = k^2 f_z.$$

Now

$$E_x = E'_x$$
$$= \frac{e}{4\pi r'^2 c}\left[c'_x + \frac{r'}{c^3}\{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'\}_x\right],$$
$$E_y = k\{E'_y + \beta H'_z\}$$
$$= \frac{ek}{4\pi r'^2 c}\left[c'_y + \frac{r'}{c^3}\left(\{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'\}_y\left\{1+\beta\frac{c'_x}{c}\right\} - \{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'\}_x \beta\frac{c'_y}{c}\right)\right],$$
$$E_z = k\{E'_z - \beta H'_y\}$$
$$= \frac{ek}{4\pi r'^2 c}\left[0 + \frac{r'}{c^3}\{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'\}_z\left\{1+\beta\frac{c'_x}{c}\right\}\right].$$

Substituting for the primed letters their values in terms of the unprimed ones, and reducing,

$$E_x = \frac{e}{4\pi r^2 k^2 c(1-\beta\cos\alpha)^3}\left[c_x - v + \frac{rk^2}{c^3}(\{\mathbf{f} \times (\mathbf{c}-\mathbf{v})\} \times \mathbf{c})_x\right],$$
$$E_y = \frac{e}{4\pi r^2 k^2 c(1-\beta\cos\alpha)^3}\left[c_y + \frac{rk^2}{c^3}(\{\mathbf{f} \times (\mathbf{c}-\mathbf{v})\} \times \mathbf{c})_y\right],$$
$$E_z = \frac{e}{4\pi r^2 k^2 c(1-\beta\cos\alpha)^3}\left[0 + \frac{rk^2}{c^3}(\{\mathbf{f} \times (\mathbf{c}-\mathbf{v})\} \times \mathbf{c})_z\right],$$

or
$$\mathbf{E} = \left[\frac{e}{4\pi r^2 k^2 c\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{c}-\mathbf{v}+\frac{rk^2}{c^3}\{\mathbf{f}\times(\mathbf{c}-\mathbf{v})\}\times\mathbf{c}\right]\right]. \quad (14)$$

H may be obtained from **H′** and **E′** in an exactly analogous manner, but it is more easily found directly from the above expression for **E**. For

$$\mathbf{H} \equiv \frac{1}{c}(\mathbf{c} \times \mathbf{E}),$$

whence

$$\mathbf{H} = \left[\frac{e}{4\pi r^2 k^2 c^2\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\left[-\mathbf{c}\times\mathbf{v}+\frac{rk^2}{c^3}\mathbf{c}\times(\{\mathbf{f}\times(\mathbf{c}-\mathbf{v})\}\times\mathbf{c})\right]\right]. \quad (15)$$

These expressions for **E** and **H**, it must be remembered, give the values of these vectors at P at the time r/c in terms of **v** and **f** at O at the time 0, **c** having the direction of the line $\overline{OP}$. In other words, all the quantities within the heavy brackets are retarded. Each expression consists of a part involving the inverse second power of the radius vector r, and a part involving the

inverse first power only. The latter depends upon the acceleration of the element of charge, and the part of the field which it determines is known as the charge's *radiation field.*

13. Retarded potentials. In differentiating expressions such as those involved in (14) and (15) account must be taken of the fact that the quantities enclosed in the heavy brackets are retarded. Let $[\psi]$ be a scalar whose value at P at a time t is given in terms of the position and velocity of a charged particle at O at a time $t-r/c$, where r is the distance $\overline{OP}$. Then, if the coördinates x, y, z of P relative to O remain unchanged,

$$d[\psi]=\left[\frac{\partial\psi}{\partial t}dt\right],$$

where, since

$$dt=[dt]+\frac{[dr]}{c}$$

$$=[dt]-\left[\frac{\mathbf{c}\cdot\mathbf{v}\,dt}{c^2}\right],$$

it follows that

$$[dt]=\left[\frac{1}{1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}}\right]dt.$$

Consequently,

$$\frac{\partial[\psi]}{\partial t}=\left[\frac{1}{1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}}\right]\left[\frac{\partial\psi}{\partial t}\right],$$

or, symbolically,

$$\frac{\partial}{\partial t}=\left[\frac{1}{1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}}\right]\left[\frac{\partial}{\partial t}\right]. \tag{16}$$

On the other hand, if x changes by an amount dx while y, z, and t remain constant,

$$d[\psi]=\left[\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial t}dt\right],$$

where

$$[dx]=dx,$$

$$[dt]=-\left[\frac{\frac{c_x}{c}}{1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}}\right]dx.$$

Hence
$$\frac{\partial[\psi]}{\partial x}=\left[\frac{\partial\psi}{\partial x}\right]-\left[\frac{\frac{c_x}{c^2}}{1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}}\right]\left[\frac{\partial\psi}{\partial t}\right],$$

and similar expressions hold for differentiation with respect to y and z. Therefore
$$\nabla=[\nabla]-\left[\frac{\frac{\mathbf{c}}{c^2}}{1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}}\right]\left[\frac{\partial}{\partial t}\right]. \tag{17}$$

Consider the retarded scalar potential function
$$[\psi]\equiv\left[\frac{1}{r\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)}\right].$$

It will be shown that **E** and **H** due to a point charge may be expressed as derivatives of this function in much the same way as the field strength due to a point charge in electrostatics may be given in terms of the gradient of the potential function $1/r$. For
$$\frac{1}{c}\frac{\partial}{\partial t}[\psi]=\left[\frac{c\left\{(1-\beta^2)-\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)+\frac{r}{c^3}\,\mathbf{c}\cdot\mathbf{f}\right\}}{r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\right],$$
$$\nabla[\psi]=\left[\frac{\mathbf{c}\left\{-(1-\beta^2)-\frac{r}{c^3}\,\mathbf{c}\cdot\mathbf{f}\right\}+\mathbf{v}\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)}{r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\right],$$
and
$$\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{c}\psi\right]=\left[\frac{\frac{r}{c^3}c^2\mathbf{f}}{r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^2}+\frac{\left\{(1-\beta^2)-\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)+\frac{r}{c^3}\,\mathbf{c}\cdot\mathbf{f}\right\}\mathbf{v}}{r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\right].$$

Therefore
$$-\nabla[\psi]-\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{c}\psi\right]=\left[\frac{(1-\beta^2)(\mathbf{c}-\mathbf{v})+\frac{r}{c^3}\left\{\mathbf{f}\times(\mathbf{c}-\mathbf{v})\right\}\times\mathbf{c}}{r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\right],$$

and comparison with (14) shows that

$$\mathbf{E} = -\frac{e}{4\pi}\nabla[\psi] - \frac{e}{4\pi c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{c}\psi\right].$$

Moreover,

$$\nabla\times\left[\frac{\mathbf{v}}{c}\psi\right] = [\psi]\,\nabla\times\left[\frac{\mathbf{v}}{c}\right] - \left[\frac{\mathbf{v}}{c}\right]\times\nabla[\psi]$$

$$= \left[\frac{-\frac{r}{c^3}\mathbf{c}\times\mathbf{f}}{r^2\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^2} - \frac{\left\{(1-\beta^2)+\frac{r}{c^3}\mathbf{c}\cdot\mathbf{f}\right\}\mathbf{c}\times\mathbf{v}}{r^2c^2\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\right]$$

$$= \left[\frac{-(1-\beta^2)\mathbf{c}\times\mathbf{v}+\frac{r}{c^3}\mathbf{c}\times(\{\mathbf{f}\times(\mathbf{c}-\mathbf{v})\}\times\mathbf{c})}{r^2c^2\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^3}\right],$$

and comparison with (15) shows that

$$\mathbf{H} = \frac{e}{4\pi}\nabla\times\left[\frac{\mathbf{v}}{c}\psi\right].$$

Whether the field is due to a single point charge or to a number of such charges, if

$$\phi \equiv \sum\frac{e}{4\pi}[\psi]$$

$$\equiv \sum\left[\frac{e}{4\pi r\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)}\right], \tag{18}$$

and

$$\mathbf{a} \equiv \sum\frac{e}{4\pi}\left[\frac{\mathbf{v}}{c}\psi\right]$$

$$\equiv \sum\left[\frac{e\mathbf{v}}{4\pi rc\left(1-\frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)}\right], \tag{19}$$

then

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\dot{\mathbf{a}}, \tag{20}$$

$$\mathbf{H} = \nabla\times\mathbf{a}. \tag{21}$$

Hence an electromagnetic field may be specified by the values at all points and times of either the two vectors **E** and **H** or the scalar potential ϕ and the vector potential **a**.

CHAPTER III

THE SIMULTANEOUS FIELD OF A POINT CHARGE

14. Constant velocity. Let a point charge e which has a constant velocity $\mathbf{v}$ relative to S be at the origin O of S at the time 0. It is desired to find the values of $\mathbf{E}$ and $\mathbf{H}$ at a point P (Fig. 5) at the *same* instant in terms of the coördinates of P and the velocity $\mathbf{v}$. Choose axes so that the velocity of the charge is along the X axis and the point P is in the XY plane. The point Q occupied by the charge at the time $-\frac{r}{c}$ is its effective position. Hence equation (14), section 12, gives for the field strength at P at the time 0,

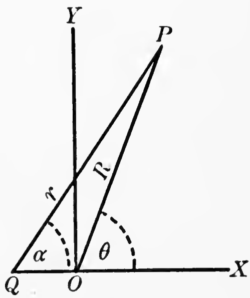

FIG. 5

$$\mathbf{E}=\left[\frac{e}{4\,\pi r^2k^2c\,(1-\beta\cos\alpha)^3}\{\mathbf{c}-\mathbf{v}\}\right].$$

Now $\qquad \overline{QO}=\beta\,r,$*

and therefore the vector $\mathbf{c}-\mathbf{v}$, and consequently $\mathbf{E}$, have the direction $\overline{OP}$.

From the geometry of the figure

$$r^2=\beta^2r^2+R^2+2\,\beta rR\cos\theta,$$

or

$$\frac{R}{r}=\sqrt{1-\beta^2\sin^2\theta}-\beta\cos\theta.$$

Hence $\frac{1}{c}|\mathbf{c}-\mathbf{v}|=\sqrt{1-\beta^2\sin^2\theta}-\beta\cos\theta.$

Moreover,

$$1-\beta\cos\alpha=\sqrt{1-\beta^2\sin^2\theta}\{\sqrt{1-\beta^2\sin^2\theta}-\beta\cos\theta\}.$$

* In order to avoid cumbersome equations in this and the succeeding article, the retarded quantity r is not enclosed in brackets.

Therefore
$$E = \frac{e}{4\pi R^2} \frac{(1-\beta^2)}{(1-\beta^2 \sin^2\theta)^{\frac{3}{2}}}. \tag{1}$$

The magnetic intensity **H** is directed upward at right angles to the plane of the figure. Its magnitude is given by

$$H = E \sin(\theta - \alpha).$$

But
$$c \sin(\theta - \alpha) = v \sin\theta.$$

Hence
$$H = \frac{e}{4\pi R^2} \frac{(1-\beta^2)\beta \sin\theta}{(1-\beta^2 \sin^2\theta)^{\frac{3}{2}}}. \tag{2}$$

The expression for E shows that the lines of force diverge radially from the moving charge (Fig. 6), but, instead of spreading out uniformly in all directions, as in the case of a static charge, they are crowded together in the equatorial belt and spread apart in the polar regions. The greater the speed the more pronounced this disparity, until, if the velocity of light is attained, the entire field is confined to the equatorial plane.

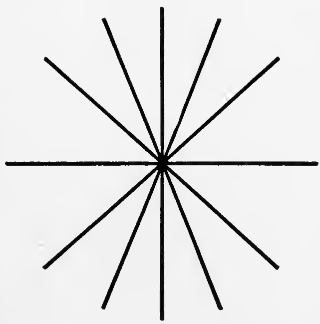

FIG. 6

If lines are drawn so as to have everywhere the direction of the vector **H**, these magnetic lines of force will be circles in planes at right angles to the line of motion with centers lying on this line. If the magnetic lines, like the electric lines, indicate by their density the magnitude of the corresponding vector, a similar crowding together in the equatorial belt and spreading apart in the polar regions will exist. However, the total number of magnetic lines of force in the field, unlike that of the electric type, will be greater the greater the speed of the charged particle.

15. Constant acceleration. Consider a point charge moving with an acceleration $\boldsymbol{\phi}$, which always has the same value relative to that system, reciprocal to S, in which the charge happens to be at rest at the instant considered. Let this charge come to rest momentarily at the origin O of S at the time 0. It is desired

to find the values of **E** and **H** at a point P (Fig. 7) at the *same* instant in terms of the coördinates of P and the acceleration $\boldsymbol{\phi}$. Choose axes so that $\boldsymbol{\phi}$ is along the X axis and P lies in the XY plane. The effective position of the charge is the point Q, which it occupied at the time $-\frac{r}{c}\cdot$ From (27), section 6,

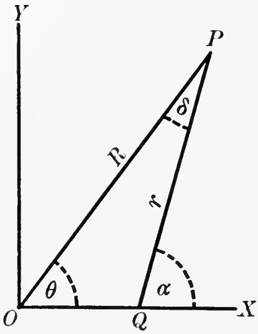

FIG. 7

$$f=\phi(1-\beta^2)^{\frac{3}{2}}.$$

Integrating, the velocity at Q is found to be given by

$$\beta=\frac{\frac{\phi r}{c^2}}{\sqrt{1+\frac{\phi^2 r^2}{c^4}}},$$

and the distance $\overline{OQ}$ by

$$\overline{OQ}=\frac{c^2}{\phi}\left\{\sqrt{1+\frac{\phi^2 r^2}{c^4}}-1\right\}.$$

If the components of **E** along and at right angles to QP are determined separately, it follows from equation (14), section 12, that

$$E_r=\left[\frac{e}{4\pi r^2 k^2}\frac{1}{(1+\beta\cos\alpha)^2}\right], \tag{3}$$

$$E_\alpha=0,$$

showing that **E** is parallel to **c**.

Now, from geometry,

$$r=R\frac{\sqrt{1+\frac{\phi R}{c^2}\cos\theta+\frac{\phi^2 R^2}{4c^4}}}{1+\frac{\phi R}{c^2}\cos\theta},$$

$$R\cos\theta=r\cos\alpha+\overline{OQ}.$$

Hence $$1+\beta\cos\alpha=\frac{1+\frac{\phi R}{c^2}\cos\theta}{\sqrt{1+\frac{\phi^2 r^2}{c^4}}}.$$

Substituting in the expression for E_r,

$$E_r = \frac{e}{4\pi R^2} \frac{1}{\left(1 + \frac{\phi R}{c^2}\cos\theta + \frac{\phi^2 R^2}{4c^4}\right)}. \tag{4}$$

To find the components of **E** along and perpendicular to $\overline{OP}$ it is necessary to obtain the values of $\cos\delta$ and $\sin\delta$. From the geometry of the figure it follows that

$$\cos\delta = \frac{1 + \frac{\phi R}{2c^2}\cos\theta}{\sqrt{1 + \frac{\phi R}{c^2}\cos\theta + \frac{\phi^2 R^2}{4c^4}}},$$

$$\sin\delta = \frac{\frac{\phi R}{2c^2}\sin\theta}{\sqrt{1 + \frac{\phi R}{c^2}\cos\theta + \frac{\phi^2 R^2}{4c^4}}};$$

whence

$$E_R = \frac{e}{4\pi R^2} \frac{1 + \frac{\phi R}{2c^2}\cos\theta}{\left(1 + \frac{\phi R}{c^2}\cos\theta + \frac{\phi^2 R^2}{4c^4}\right)^{\frac{3}{2}}}, \tag{5}$$

$$E_\theta = \frac{e}{4\pi R^2} \frac{\frac{\phi R}{2c^2}\sin\theta}{\left(1 + \frac{\phi R}{c^2}\cos\theta + \frac{\phi^2 R^2}{4c^4}\right)^{\frac{3}{2}}}. \tag{6}$$

To obtain the equation of the lines of force, use may be made of the relation

$$R\frac{d\theta}{dR} = \tan\delta$$

$$= \frac{\frac{\phi R}{2c^2}\sin\theta}{1 + \frac{\phi R}{2c^2}\cos\theta}.$$

The solution of this differential equation is

$$\cot\theta + \frac{\phi R}{2c^2}\frac{1}{\sin\theta} = b,$$

where b is the constant of integration. Writing this equation in the form

$$\left(R\cos\theta+\frac{c^2}{\phi}\right)^2+(R\sin\theta-b)^2=b^2+\frac{c^4}{\phi^2}, \tag{7}$$

it is seen that the lines of force are circles passing through O with centers in a plane at right angles to the X axis and at a distance $-c^2/\phi$ from O. Fig. 8 shows the section of the field cut by a plane through the X axis. The full lines represent the lines of force.

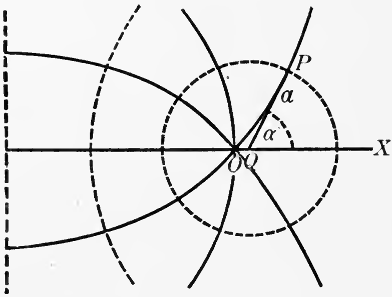

FIG. 8

Consider an electric field all parts of which happen to be at rest in a single system S at the same time. A surface in S, so constructed as to be everywhere normal to the lines of force, is called a *level surface*. For a field permanently at rest in S level surfaces are identical with the equipotential surfaces of electrostatics. In the case under discussion of a charged particle moving with constant acceleration, **E** has been shown to be parallel to **c** simultaneously at all points. Hence all parts of the field are at rest at the same time and consequently level surfaces can be constructed. The differential equation of these surfaces is,

$$R\frac{d\theta}{dR}=-\cot\delta$$

$$=-\frac{1+\dfrac{\phi R}{2c^2}\cos\theta}{\dfrac{\phi R}{2c^2}\sin\theta},$$

of which the solution is

$$\left\{R\cos\theta-\left(h-\frac{c^2}{\phi}\right)\right\}^2+\{R\sin\theta\}^2=h^2-\frac{c^4}{\phi^2}, \tag{8}$$

where h is the constant of integration. This is the equation of a family of spheres with centers at the effective positions of the moving charge. Their traces are shown by broken lines in Fig. 8.

Since **E** is parallel to **c** at the instant considered, **H** is everywhere zero.

16. General case. The retarded expressions for **E** and **H** deduced in the preceding chapter show that the field at a point P and time 0 is conditioned by the motion of the charge producing that field at a time $-\frac{[r]}{c}$, where $[r]$ is the distance of P from the effective position of the moving particle. Therefore the specification of the entire field at the time 0 involves the complete past history of the charged particle. Since, for physical reasons, the motion of this particle must be continuous, the past history of its motion is contained in its present position, velocity, acceleration, and higher time derivatives of the positional vector. Hence the simultaneous values of **E** and **H** may be expressed as series in these quantities. While these series may fail to converge for distant portions of the field, or for very rapidly changing motion, their form will make evident their very rapid convergence for all cases to which they will be applied.

If the point at which **E** and **H** are to be evaluated is chosen as origin, and if $[x]$, $[y]$, $[z]$ are the coördinates of the effective position of the charged particle relative to this origin,

$$\frac{[c_x]}{c} = -\left[\frac{x}{r}\right],$$

etc., and the x component of the retarded expression (14), section 12, for the field strength, may be written

$$E_x = \left[\frac{e}{4\pi r^2}\left(1+\frac{\mathbf{v}\cdot\mathbf{r}}{cr}\right)^{-3}\left\{\left(1-\frac{v^2}{c^2}-\frac{\mathbf{f}\cdot\mathbf{r}}{c^2}\right)\left(-\frac{x}{r}-\frac{v_x}{c}\right)-\frac{f_x r}{c^2}\left(1+\frac{\mathbf{v}\cdot\mathbf{r}}{cr}\right)\right\}\right]. \quad (9)$$

If x, y, z are the coördinates of the simultaneous position of the charge, and $\mathbf{v}$, $\mathbf{f}$, $\dot{\mathbf{f}} \cdots$ its simultaneous velocity, acceleration, etc., then

$$[x] = x - v_x\frac{[r]}{c} + \frac{1}{2}f_x\frac{[r^2]}{c^2} - \frac{1}{6}\dot{f}_x\frac{[r^3]}{c^3} + \frac{1}{24}\ddot{f}_x\frac{[r^4]}{c^4} - \frac{1}{120}\dddot{f}_x\frac{[r^5]}{c^5}\cdots,$$

$$[v_x] = v_x - f_x\frac{[r]}{c} + \frac{1}{2}\dot{f}_x\frac{[r^2]}{c^2} - \frac{1}{6}\ddot{f}_x\frac{[r^3]}{c^3} + \frac{1}{24}\dddot{f}_x\frac{[r^4]}{c^4}\cdots,$$

$$[f_x] = f_x - \dot{f}_x\frac{[r]}{c} + \frac{1}{2}\ddot{f}_x\frac{[r^2]}{c^2} - \frac{1}{6}\dddot{f}_x\frac{[r^3]}{c^3}\cdots,$$

and similar expressions hold for the y and z components. Put

$$m_x \equiv \frac{x}{r},\ \beta_x \equiv \frac{v_x}{c},\ \gamma_x \equiv \frac{f_x r}{c^2},\ \delta_x \equiv \frac{\dot{f}_x r^2}{c^3},\ \epsilon_x \equiv \frac{\ddot{f}_x r^3}{c^4},\ \zeta_x \equiv \frac{\dddot{f}_x r^4}{c^5}.$$

Then

$$[r^2]=r^2\Big\{1-2\,\boldsymbol{\beta}\cdot\mathbf{m}\frac{[r]}{r}+(\boldsymbol{\gamma}\cdot\mathbf{m}+\beta^2)\frac{[r^2]}{r^2}-\frac{1}{3}(\boldsymbol{\delta}\cdot\mathbf{m}+3\,\boldsymbol{\gamma}\cdot\boldsymbol{\beta})\frac{[r^3]}{r^3}$$
$$+\frac{1}{12}(\boldsymbol{\epsilon}\cdot\mathbf{m}+4\,\boldsymbol{\delta}\cdot\boldsymbol{\beta}+3\gamma^2)\frac{[r^4]}{r^4}$$
$$-\frac{1}{60}(\boldsymbol{\zeta}\cdot\mathbf{m}+5\,\boldsymbol{\epsilon}\cdot\boldsymbol{\beta}+10\,\boldsymbol{\delta}\cdot\boldsymbol{\gamma})\frac{[r^5]}{r^5}\cdots\Big\}.$$

Now put

$$\tau\equiv\frac{[r]}{rk},$$
$$a\equiv\boldsymbol{\beta}\cdot\mathbf{m}k,$$
$$b\equiv\boldsymbol{\gamma}\cdot\mathbf{m}k^2,$$
$$c\equiv(\boldsymbol{\delta}\cdot\mathbf{m}+3\,\boldsymbol{\gamma}\cdot\boldsymbol{\beta})k^3,$$
$$d\equiv(\boldsymbol{\epsilon}\cdot\mathbf{m}+4\,\boldsymbol{\delta}\cdot\boldsymbol{\beta}+3\,\gamma^2)k^4,$$
$$e\equiv(\boldsymbol{\zeta}\cdot\mathbf{m}+5\,\boldsymbol{\epsilon}\cdot\boldsymbol{\beta}+10\,\boldsymbol{\delta}\cdot\boldsymbol{\gamma})k^5,$$

whence the previous equation becomes simply

$$\tau^2=1-2\,a\tau+b\tau^2-\tfrac{1}{3}c\tau^3+\tfrac{1}{12}d\tau^4-\tfrac{1}{60}e\tau^5\cdots, \tag{10}$$

and

$$\left[1+\frac{\mathbf{v}\cdot\mathbf{r}}{cr}\right]=\frac{1}{k^2\tau^2}\Big\{1-a\tau+0+\frac{1}{6}c\tau^3-\frac{1}{12}d\tau^4+\frac{1}{40}e\tau^5\cdots\Big\}$$
$$\equiv\frac{I}{k^2\tau^2}.$$

After some reduction it is found that

$$\left[\left(1-\frac{v^2}{c^2}-\frac{\mathbf{f}\cdot\mathbf{r}}{c^2}\right)\left(-\frac{x}{r}-\frac{v_x}{c}\right)-\frac{f_x r}{c^2}\left(1+\frac{\mathbf{v}\cdot\mathbf{r}}{cr}\right)\right]$$
$$=-\frac{1}{k^3\tau^3}\Big\{m_x\left(1-2\,a\tau+0+\frac{4}{6}c\tau^3-\frac{5}{12}d\tau^4+\frac{6}{40}e\tau^5\cdots\right)$$
$$+0$$
$$-\frac{1}{2}\gamma_x k^2\tau^2\left(-1+0+0+\frac{2}{6}c\tau^3-\frac{3}{12}d\tau^4+\frac{4}{40}e\tau^5\cdots\right)$$
$$+\frac{1}{3}\delta_x k^3\tau^3\left(-2+a\tau+0+\frac{1}{6}c\tau^3-\frac{2}{12}d\tau^4+\frac{3}{40}e\tau^5\cdots\right)$$
$$-\frac{1}{8}\epsilon_x k^4\tau^4\left(-3+2\,a\tau+0+0-\frac{1}{12}d\tau^4+\frac{2}{40}e\tau^5\cdots\right)$$
$$+\frac{1}{30}\zeta_x k^5\tau^5\left(-4+3\,a\tau+0-\frac{1}{6}c\tau^3+0+\frac{1}{40}e\tau^5\cdots\right)\cdots\Big\}$$
$$\equiv-\frac{J}{k^3\tau^3}.$$

Hence
$$E_x = -\frac{e}{4\pi r^2} k\tau I^{-3} J. \tag{11}$$

Returning to (10) and solving for τ by successive approximations, it is found that

$$\begin{aligned}\tau = 1 - a(1 - \tfrac{1}{2}a + 0 + \tfrac{1}{8}a^3 + 0)\\ + \tfrac{1}{2}b(1 - 2a + \tfrac{3}{2}a^2 + 0) + \tfrac{3}{8}b^2(1 - \tfrac{8}{3}a)\\ - \tfrac{1}{6}c(1 - 3a + 4a^2) - \tfrac{1}{3}bc + \tfrac{1}{24}d(1 - 4a) - \tfrac{1}{120}e \cdots.\end{aligned} \tag{12}$$

Substituting this value of τ in (11) and reducing, the following expression is obtained for the X component of the electric intensity,

$$\begin{aligned}E_x = -\frac{ek}{4\pi r^2}\Big\{m_x\Big(1 - \frac{3}{2}a^2 + \frac{15}{8}a^4 \cdots + \frac{1}{2}b - \frac{9}{4}a^2 b \cdots\\ + \frac{3}{8}b^2 + \frac{1}{2}ac - \frac{1}{8}d + \frac{1}{15}e \cdots\Big) + \frac{1}{2}\gamma_x k^2\Big(1 - \frac{3}{2}a^2 + \frac{3}{2}b - \frac{4}{3}c \cdots\Big)\\ - \frac{2}{3}\delta_x k^3\Big(1 - \frac{3}{2}a + 2b \cdots\Big) + \frac{3}{8}\epsilon_x k^4\Big(1 - \frac{8}{3}a \cdots\Big) - \frac{4}{30}\zeta_x k^5 \cdots\Big\}.\end{aligned} \tag{13}$$

The X component of the magnetic intensity may be obtained most easily from the relation

$$H_x = -\left[\frac{y}{r}\right]E_z + \left[\frac{z}{r}\right]E_y,$$

which gives, after some reduction,

$$\begin{aligned}H_x = -\frac{ek}{4\pi r^2}\Big\{(\beta_y m_z - \beta_z m_y)\Big(1 - \frac{3}{2}a^2 + \frac{1}{2}b \cdots\Big)\\ + ak(\gamma_y m_z - \gamma_z m_y) - \frac{1}{2}k^2(\delta_y m_z - \delta_z m_y) + \frac{1}{3}k^3(\epsilon_y m_z - \epsilon_z m_y)\\ - \frac{1}{2}k^2(\gamma_y \beta_z - \gamma_z \beta_y) + \frac{2}{3}k^3(\delta_y \beta_z - \delta_z \beta_y) \cdots\Big\}.\end{aligned} \tag{14}$$

Equations (13) and (14) give **E** and **H** at the origin in terms of the simultaneous coördinates, velocity, acceleration, etc. of the charge e. To obtain the values of **E** and **H** at a point x, y, z

due to a charge e at the origin, the signs of the coördinates in these two equations must be changed, giving

$$E_x = \frac{ek}{4\pi r^2}\Big\{m_x\Big(1-\frac{3}{2}a_1^2+\frac{15}{8}a_1^4\cdots-\frac{1}{2}b_1+\frac{9}{4}a_1^2b_1\cdots+\frac{3}{8}b_1^2$$

$$+\frac{1}{2}a_1c_1+\frac{1}{8}d_1-\frac{1}{15}e_1\cdots\Big)-\frac{1}{2}\gamma_x k^2\Big(1-\frac{3}{2}a_1^2-\frac{3}{2}b_1+\frac{4}{3}c_1\cdots\Big)$$

$$+\frac{2}{3}\delta_x k^3\Big(1+\frac{3}{2}a_1-2b_1\cdots\Big)-\frac{3}{8}\epsilon_x k^4\Big(1+\frac{8}{3}a_1\cdots\Big)+\frac{4}{30}\zeta_x k^5\cdots\Big\}, \quad (15)$$

$$H_x = \frac{ek}{4\pi r^2}\Big\{(\beta_y m_z-\beta_z m_y)\Big(1-\frac{3}{2}a_1^2-\frac{1}{2}b_1\cdots\Big)$$

$$-a_1k(\gamma_y m_z-\gamma_z m_y)-\frac{1}{2}k^2(\delta_y m_z-\delta_z m_y)+\frac{1}{3}k^3(\epsilon_y m_z-\epsilon_z m_y)$$

$$+\frac{1}{2}k^2(\gamma_y\beta_z-\gamma_z\beta_y)-\frac{2}{3}k^3(\delta_y\beta_z-\delta_z\beta_y)\cdots\Big\}, \quad (16)$$

where

$$a_1 \equiv \boldsymbol{\beta}\cdot\mathbf{m}k,$$
$$b_1 \equiv \boldsymbol{\gamma}\cdot\mathbf{m}k^2,$$
$$c_1 \equiv (\boldsymbol{\delta}\cdot\mathbf{m}-3\,\boldsymbol{\gamma}\cdot\boldsymbol{\beta})k^3,$$
$$d_1 \equiv (\boldsymbol{\epsilon}\cdot\mathbf{m}-4\,\boldsymbol{\delta}\cdot\boldsymbol{\beta}-3\,\gamma^2)k^4,$$
$$e_1 \equiv (\boldsymbol{\zeta}\cdot\mathbf{m}-5\,\boldsymbol{\epsilon}\cdot\boldsymbol{\beta}-10\,\boldsymbol{\delta}\cdot\boldsymbol{\gamma})k^5.$$

CHAPTER IV

THE DYNAMICAL EQUATION OF AN ELECTRON

17. Electrical theory of matter. All matter will be assumed to be made up of positive and negative electrons. An electron will be defined as an invariable charge, of magnitude approximately

$$4.77(10)^{-10}\sqrt{4\pi}$$

Heaviside-Lorentz units, distributed over a surface which is spherical in form to an observer in that system, reciprocal to S, in which the electron happens to be momentarily at rest. A positive electron will be considered to differ from a negative one only in the sign of the charge involved and the radius of the spherical surface over which it is distributed.

The *electromagnetic force* $d\mathbf{K}$ on an element of charge de, as measured in that system, reciprocal to S, in which this charge happens to be at rest at the instant considered, is defined as the product of the field strength $\mathbf{E}$ by the charge de. The extension of this definition to the case of a system in which de is not at rest, will be given in the next section.

The distribution of charge on the surface of an electron will be supposed to be such as to make the tangential force due to its own field zero at all points of the surface, all measurements being made in that system relative to which the electron is momentarily at rest. This assumption is introduced merely for the purpose of simplifying the analysis (section 20) involved in determining the dynamical equation of an electron moving with constant acceleration. To the number of terms to which the analysis is carried in the general case (section 21), no change in the dynamical equation is introduced if this hypothesis is replaced by the more probable assumption that the distribution of charge is such as to make the tangential force due to the

total field, that is, the resultant of the impressed field and the electron's field, equal to zero, or by the simple assumption that the distribution of charge is always uniform.

18. Dynamical assumption. In the previous chapters the discussion has been concerned with the determination of the field of a charged particle. It must be borne in mind, however, that the lines of force constituting such a field are nothing more than convenient geometrical representations to be employed in describing the effect of one charged particle on another, and that no reason exists for attributing a greater substantiality to them than to any other arbitrary convention, such as, for instance, parallels of latitude on the earth's surface. The representation of a field by lines of force has led to the concept of electric intensity, and the electromagnetic force on an element of charge, as measured in the system in which the charge is momentarily at rest, has been defined in terms of this quantity. In order to pass from these definitions to the quantitative description of the effect of one electron on another, it is necessary to introduce the following dynamical assumption:

The motion of an electron is such as to make the total electromagnetic force on it, as measured in that system, reciprocal to S, in which it happens to be momentarily at rest, equal to zero. By the total electromagnetic force is to be understood the resultant of the force due to the impressed field and that due to the charge's own field. With forces which are not electrical in nature, such as must exist if a dynamical explanation of the stability of the electron is possible, the present discussion is not concerned. While extra-electrical stresses on a single electron may be of great intensity, their resultant will be assumed to be always zero. Moreover, such forces will be supposed to be comparatively negligible when the effect of one electron on another is under consideration. Thus no account will be taken of the gravitational attraction between two electrons, as it will be deemed quite unimportant compared to the electrical attraction or repulsion.

Consider an electron which is at rest in S' at the instant considered. Then the dynamical assumption stated above requires that

$$\int \mathbf{E}' de = 0. \tag{1}$$

Substituting for the components of $\mathbf{E}'$ their values in terms of the components of $\mathbf{E}$ and $\mathbf{H}$, as given in equations (1), (2), (3), section 10, it is found that

$$\int E_x de = 0,$$

$$\int \left\{ E_y + \frac{1}{c} (\mathbf{v} \times \mathbf{H})_y \right\} de = 0,$$

$$\int \left\{ E_z + \frac{1}{c} (\mathbf{v} \times \mathbf{H})_z \right\} de = 0.$$

Hence it is natural to extend the definition of electromagnetic force given in the preceding section so as to read:

The *electromagnetic force* $d\mathbf{K}$ on an element of charge de, as measured in a system, reciprocal to S, relative to which the charge has the velocity $\mathbf{v}$ at the instant considered, is defined by

$$d\mathbf{K} \equiv \left\{ \mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{H}) \right\} de. \tag{2}$$

Then the dynamical assumption may be stated in the more general form:

The motion of an electron is such as to make the total electromagnetic force on it, as measured in any system reciprocal to S, equal to zero. Thus the dynamical equation of an electron may be found directly for any system, no matter whether the electron is at rest in that system or in motion with respect to it. However, in order to avoid unnecessary analysis, the method pursued will be first to deduce this equation relative to that system in which the electron is momentarily at rest, and then to extend it to other systems by means of the transformations already derived.

19. Constant velocity. Consider an electron permanently at rest in S'. Relative to an observer in this system the electron is a uniformly charged spherical surface of radius a with a uniform

radial field. To an observer in S, however, this electron has a constant velocity v along the X axis, and the transformation equation (18), section 6, shows that its dimensions in the direction of motion are shorter in the ratio $1:k$ when viewed from this system, while those at right angles to this direction are unchanged. Hence to an observer relative to whom an electron is moving its surface is that of an oblate spheroid with the short axis in the direction of motion.

Describe two right circular cones with vertices at the center O' of the electron and axes along the X axis such that elements of the cones make angles θ' and $\theta'+d\theta'$ respectively with their common axis. If e is the charge on the electron, the number of tubes included between the cones is

$$dN=\frac{e}{2}\sin\theta' d\theta'.$$

But
$$\cos\theta'=\frac{\cos\theta}{\sqrt{1-\beta^2\sin^2\theta}},$$

$$\sin\theta' d\theta'=\frac{(1-\beta^2)\sin\theta\, d\theta}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}.$$

Hence
$$dN=\frac{e}{2}\frac{(1-\beta^2)\sin\theta\, d\theta}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}},$$

and the electric intensity in S at any distance R from O' is

$$E=\frac{e}{4\pi R^2}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}. \tag{3}$$

The magnetic intensity is given by

$$H\equiv\frac{1}{c}|\mathbf{c}\times\mathbf{E}|$$
$$=E\beta\sin\theta,$$

and substituting for E its value from (3),

$$H=\frac{e}{4\pi R^2}\frac{(1-\beta^2)\beta\sin\theta}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}. \tag{4}$$

Comparison of these expressions with (1) and (2), section 14, shows that the external field of an electron moving with constant velocity is the same as that of an equal charge located at its center. It follows from symmetry that the resultant force on the electron due to its own field is zero. Hence the dynamical assumption requires that the impressed force shall be zero as well.

20. Constant acceleration. Consider an electron each point of which moves with an acceleration which always has the same value relative to that system, reciprocal to S, in which this point happens to be at rest at the instant considered. Let ϕ be the value of this acceleration for the point O of the electron. Choose axes so that ϕ is along the X axis. Then (27), section 6, gives for the acceleration f of this point relative to S at any time

$$f = \phi(1-\beta^2)^{\frac{3}{2}}. \tag{5}$$

Integrating, the velocity of O relative to S is found to be given by

$$\beta = \frac{\frac{\phi t}{c}}{\sqrt{1+\frac{\phi^2 t^2}{c^2}}}, \tag{6}$$

and the displacement by

$$x = \frac{c^2}{\phi}\left\{\sqrt{1+\frac{\phi^2 t^2}{c^2}}-1\right\}. \tag{7}$$

Consider a neighboring point P of the electron such that $\overline{OP}$ is parallel to the X axis and equal to $d\lambda$ when O is at rest in S. Then, since O and P have constant accelerations relative to the systems, reciprocal to S, in which they happen to be at rest at the instant considered, the principle of relativity requires that the length $\overline{OP}$ as measured by an observer in any system reciprocal to S when O is at rest in that system shall be the same as the length $\overline{OP}$ as measured by an observer in S when O is at rest in S. Hence

$$\overline{OP} = d\lambda\sqrt{1-\beta^2} \tag{8}$$

is the distance $\overline{OP}$ as measured in S when O has the velocity βc.

Now, when O is at rest in S, P may have a velocity $d\mu$ in the X direction. But the principle of relativity requires that $d\mu$

shall be the same relative to any other system, reciprocal to S, at the instant when O happens to be at rest in that system. Hence, adding to $d\lambda$ the difference between the displacements of P and O in the time t,

$$\overline{OP} = d\lambda + \frac{c^2 d\phi}{\phi^2}\{1-\sqrt{1-\beta^2}\} + \beta\frac{c\,d\mu}{\phi} \tag{9}$$

is found to be the distance $\overline{OP}$ as measured in S when O has the velocity βc.

Equating coefficients of like powers of β in the identical expressions (8) and (9), it is seen that

$$d\mu = 0,$$
$$d\phi = -\frac{\phi^2 d\lambda}{c^2}.$$

The first of these equations shows that *when one point of the electron is at rest in S, every other point is likewise at rest.* Integrating the second,

$$\phi = \frac{\phi_0}{1+\dfrac{x\phi_0}{c^2}}, \tag{10}$$

where ϕ_0 is the acceleration of O. This equation shows that points on the forward side of the electron have smaller accelerations than those on the rear. Such a difference is obviously necessary in order to produce the progressive contraction of the electron required by the principle of relativity as its velocity relative to S increases.

Obviously, the relations just obtained between the velocities and accelerations of points of the electron under consideration apply equally well to points of the field of Fig. 8, p. 37. Hence any one of the level surfaces of this field, such as that upon which the point P lies, may be considered to constitute the surface of the electron. As the charge is distributed entirely on this surface, it is necessary, in order that the external field should be the same as that due to an equal charge at O, that the density of charge should be everywhere equal to the electric intensity just outside this surface.

If e is the electron's charge, and $\mathbf{E}_1$ the strength of the external field, the impressed force is

$$\mathbf{K}_1 = e\mathbf{E}_1. \tag{11}$$

In Fig. 8, Q is the geometrical center of the electron and a its radius. The electric intensity E_2 at its surface due to an equal charge at O is given by

$$E_2 = \frac{e}{4\,\pi a^2 k^2}\frac{1}{(1+\beta\cos\alpha)^2}, \tag{12}$$

where
$$\beta = \frac{\dfrac{\phi_0 a}{c^2}}{\sqrt{1+\dfrac{\phi_0^2 a^2}{c^4}}},$$

ϕ_0 being the acceleration of the point O. As the field due to the electron vanishes everywhere within its interior, the resultant force $\mathbf{K}_2$ on this charged particle due to its own field is

$$K_2 = \tfrac{1}{2}\int E^2 \cos\alpha\, d\sigma,$$

where $d\sigma$ is an element of the surface. Substituting for E its value in terms of α and integrating,

$$\mathbf{K}_2 = -\frac{e^2\boldsymbol{\phi}_0}{6\,\pi a c^2}\sqrt{1+\frac{\phi_0^2 a^2}{c^4}}. \tag{13}$$

The point on the axis of symmetry of the electron through which a perpendicular plane would divide its surface into parts having equal charges, will be called the *center of charge.* If ϕ is the acceleration of this point,

$$\boldsymbol{\phi} = \boldsymbol{\phi}_0\sqrt{1+\frac{\phi_0^2 a^2}{c^4}}.$$

Hence
$$\mathbf{K}_2 = -\frac{e^2}{6\,\pi a c^2}\boldsymbol{\phi}. \tag{14}$$

The rest mass m is defined by

$$m \equiv \frac{e^2}{6\,\pi a c^2}.$$

Hence, as the dynamical assumption requires that

$$\mathbf{K}_1 + \mathbf{K}_2 = 0,$$

it follows that the acceleration of the center of charge is determined by

$$e\mathbf{E}_1 = m\mathbf{f} \tag{15}$$

at the instant that the electron is at rest in S.

Consider an electron which has the type of motion under discussion, and which is at rest in S' at the instant considered. Let the acceleration $\mathbf{f}'$ of the center of charge make an angle with the direction of S''s velocity relative to S. Then, dropping the subscript,

$$e\mathbf{E}' = m\mathbf{f}'.$$

Substituting in each of the component equations the values of E'_x, E'_y, E'_z from (1), (2), (3), section 10, and those of f'_x, f'_y, f'_z from (27), (28), (29), section 6, it is found that

$$eE_x = mk^3 f_x, \tag{16}$$

$$e\left\{E_y + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_y\right\} = mkf_y, \tag{17}$$

$$e\left\{E_z + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_z\right\} = mkf_z. \tag{18}$$

As the electromagnetic force in S is defined by

$$\mathbf{K} \equiv e\left\{\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{H})\right\},$$

it is seen to be necessary to distinguish between the longitudinal mass

$$m_l \equiv mk^3, \tag{19}$$

and the transverse mass

$$m_t \equiv mk. \tag{20}$$

Both masses increase with the velocity, becoming infinite as the velocity of light is approached. In terms of the transverse mass the dynamical equation may be written in the compact form

$$\mathbf{K} = \frac{d}{dt}(m_t\mathbf{v}), \tag{21}$$

where $\mathbf{K}$ is the impressed force.

21. General case. Consider an electron a point P of which is at rest in S at the time 0. Denote by $\mathbf{f}$, $\dot{\mathbf{f}}$, etc. the acceleration, rate of change of acceleration, etc. of this point. Choose axes so that the X axis has the direction of $\mathbf{f}$. Then if Q is a neighboring point of the electron whose coördinates relative to P are dx, dy, dz at the time 0, the values of these coördinates at the time dt will be

$$dx_t = dx + \left(\frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz\right)dt$$
$$+ \frac{1}{2}\left(\frac{\partial f_x}{\partial x}dx + \frac{\partial f_x}{\partial y}dy + \frac{\partial f_x}{\partial z}dz\right)dt^2 \cdots,$$

and similar expressions for dy_t and dz_t. But, as $f_x = f$, $f_y = 0$, $f_z = 0$,

$$dx_t = dx\sqrt{1 - \frac{f_x^2 dt^2}{c^2}}$$
$$= dx\left(1 - \frac{1}{2}\frac{f_x^2}{c^2}dt^2 \cdots\right),$$
$$dy_t = dy,$$
$$dz_t = dz.$$

Equating coefficients of like powers of dt and dx, dy, dz in the equivalent expressions for dx_t, dy_t, dz_t, it is found that

$$\begin{aligned} &\frac{dv_x}{\partial x} = \frac{\partial v_x}{\partial y} = \frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial x} = \text{etc.} = 0, \\ &\frac{\partial f_x}{\partial x} = -\frac{f_x^2}{c^2}, \\ &\frac{\partial f_x}{\partial y} = \frac{\partial f_x}{\partial z} = \frac{\partial f_y}{\partial x} = \text{etc.} = 0. \end{aligned} \tag{22}$$

Hence the velocity is not a function of the coördinates, and *when one point of the electron is at rest in S, all other points are also at rest.* Moreover, the y and z components of the acceleration are not functions of the coördinates, and the x component is a function of x only.

Before proceeding further, it is convenient to distinguish between the orders of possible factors which may appear in the

dynamical equation of the electron. If a denotes the radius of the electron, it will be considered that

$$\beta \text{ is of the first order,}$$

$$\frac{fa}{c^2} \text{ is of the second order,}$$

$$\frac{\dot{f}a^2}{c^3} \text{ is of the third order,}$$

$$\frac{\ddot{f}a^3}{c^4} \text{ is of the fourth order, and}$$

$$\frac{\dddot{f}a^4}{c^5} \text{ is of the fifth order.}$$

The dynamical equation to be obtained will be carried through the fifth order as thus defined.

As before, the impressed force on the electron is

$$\mathbf{K}_1 = e\,\mathbf{E}_1. \tag{23}$$

The next step is to evaluate the reaction on the electron of its own field. Let the origin be located at a point O on the surface of the electron, and for the purposes of the following analysis let the orientation of the axes relative to $\mathbf{f}$ be arbitrary. Then if P is another point on the surface of the electron whose coördinates relative to O are x, y, z, equation (15), section 16, gives for the force exerted by a charge de at O on a charge de_1 at P

$$\begin{aligned} dK_{2x} = \frac{de\,de_1}{4\pi r^2}\Big\{ & m_x\Big(1 - \frac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{m} + \frac{3}{8}\overline{\boldsymbol{\gamma}\cdot\mathbf{m}}^2 + \frac{1}{8}\boldsymbol{\epsilon}\cdot\mathbf{m} - \frac{3}{8}\gamma^2 \\ & - \frac{1}{15}\boldsymbol{\zeta}\cdot\mathbf{m} + \frac{2}{3}\boldsymbol{\delta}\cdot\boldsymbol{\gamma}\cdots\Big) \\ & - \frac{1}{2}\gamma_x\Big(1 - \frac{3}{2}\boldsymbol{\gamma}\cdot\mathbf{m} + \frac{4}{3}\boldsymbol{\delta}\cdot\mathbf{m}\cdots\Big) + \frac{2}{3}\delta_x(1 - 2\,\boldsymbol{\gamma}\cdot\mathbf{m}\cdots) \\ & - \frac{3}{8}\epsilon_x + \frac{4}{30}\zeta_x\cdots\Big\}. \end{aligned} \tag{24}$$

Integration of this expression with respect to de_1 will give the force exerted on the rest of the electron by the charge de at O. Finally, on integrating with respect to de, the X component of the force on the electron due to the reaction of its own field

will be obtained. In performing these integrations, the charge on the electron may be considered to be uniformly distributed over its surface, for, even under the conditions assumed in the last section, reference to (12) shows that the divergence from uniformity there implied leads to no term of less than sixth order which does not vanish upon integration. Moreover, it is unnecessary to take into account the variation of **f** from point to point of the electron, since (22) shows that the only term involved of less than sixth order vanishes when the integration is performed. *A fortiori* the variations of the derivatives of **f** are negligible.

Omitting from the integrand all terms which vanish on integration, (21) leads to the expression

$$K_{2x} = \frac{1}{4\pi}\iint de\,de_1\left\{-\frac{1}{2}\frac{f_x}{c^2 r}\left(1+\frac{x^2}{r^2}\right)+\frac{2}{3}\frac{\dot{f}_x}{c^3}-\frac{1}{8}\frac{\ddot{f}_x r}{c^4}\left(3-\frac{x^2}{r^2}\right)\right.$$

$$\left.+\frac{1}{30}\frac{\dddot{f}_x r^2}{c^5}\left(4-2\frac{x^2}{r^2}\right)\cdots\right\}.$$

Since

$$\iint r^m de\,de_1 = 3\iint \frac{x^2}{r^2} r^m de\,de_1 = \frac{2^{m+1}}{m+2}a^m e^2,$$

$$K_{2x} = -\frac{e^2}{6\pi ac^2}f_x + \frac{e^2}{6\pi c^3}\dot{f}_x - \frac{e^2 a}{9\pi c^4}\ddot{f}_x + \frac{e^2 a^2}{18\pi c^5}\dddot{f}_x \cdots, \tag{25}$$

and similar expressions hold for the y and z components. Hence the dynamical assumption leads to the vector equation

$$e\mathbf{E}_1 = \frac{e^2}{6\pi ac^2}\mathbf{f} - \frac{e^2}{6\pi c^3}\dot{\mathbf{f}} + \frac{e^2 a}{9\pi c^4}\ddot{\mathbf{f}} - \frac{e^2 a^2}{18\pi c^5}\dddot{\mathbf{f}}\cdots. \tag{26}$$

Consider an electron which is momentarily at rest in S'. Then, dropping the subscript,

$$e\mathbf{E}' = \frac{e^2}{6\pi ac^2}\mathbf{f}' - \frac{e^2}{6\pi c^3}\dot{\mathbf{f}}' + \frac{e^2 a}{9\pi c^4}\ddot{\mathbf{f}}' - \frac{e^2 a^2}{18\pi c^5}\dddot{\mathbf{f}}' \cdots.$$

Equations (27), (28), (29), section 6, give

$$f'_x = k^3 f_x,$$

$$f'_y = k^2 f_y,$$

$$f'_z = k^2 f_z.$$

Differentiating (24), (25), (26), section 6, once, twice, and thrice, with respect to t', and then placing $\mathbf{V}'$ equal to zero, it is found that

$$\dot{f}'_x = k^4\dot{f}_x + 3\,k^6\mathbf{f}\cdot\boldsymbol{\beta}\,\frac{f_x}{c},$$

$$\dot{f}'_y = k^3\dot{f}_y + 3\,k^5\mathbf{f}\cdot\boldsymbol{\beta}\,\frac{f_y}{c},$$

$$\dot{f}'_z = k^3\dot{f}_z + 3\,k^5\mathbf{f}\cdot\boldsymbol{\beta}\,\frac{f_z}{c};$$

$$\ddot{f}'_x = k^5\ddot{f}_x + \text{terms of sixth and higher orders},$$

$$\ddot{f}'_y = k^4\ddot{f}_y + \cdots,$$

$$\ddot{f}'_z = k^4\ddot{f}_z + \cdots;$$

$$\dddot{f}'_x = k^6\,\dddot{f}_x + \cdots,$$

$$\dddot{f}'_y = k^5\dddot{f}_y + \cdots,$$

$$\dddot{f}'_z = k^5\dddot{f}_z + \cdots.$$

Substituting in each component of the dynamical equation these values of f'_x, f'_y, f'_z, and their derivatives, as well as the values of E'_x, E'_y, E'_z, from (1), (2), (3), section 10, the dynamical equation of an electron for a system relative to which it has a velocity v is found to have the following components:

$$eE_x = \frac{e^2k^3}{6\,\pi ac^2}f_x - \frac{e^2k^6}{2\,\pi c^4}\mathbf{f}\cdot\boldsymbol{\beta}f_x \cdots$$

$$- \frac{e^2k^4}{6\,\pi c^3}\dot{f}_x + \frac{e^2ak^5}{9\,\pi c^4}\ddot{f}_x - \frac{e^2a^2k^6}{18\,\pi c^5}\dddot{f}_x \cdots, \tag{27}$$

$$e\left\{E_y + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_y\right\} = \frac{e^2k}{6\,\pi ac^2}f_y - \frac{e^2k^4}{2\,\pi c^4}\mathbf{f}\cdot\boldsymbol{\beta}f_y \cdots$$

$$- \frac{e^2k^2}{6\,\pi c^3}\dot{f}_y + \frac{e^2ak^3}{9\,\pi c^4}\ddot{f}_y - \frac{e^2a^2k^4}{18\,\pi c^5}\dddot{f}_y \cdots, \tag{28}$$

$$e\left\{E_z + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_z\right\} = \frac{e^2k}{6\,\pi ac^2}f_z - \frac{e^2k^4}{2\,\pi c^4}\mathbf{f}\cdot\boldsymbol{\beta}f_z \cdots$$

$$- \frac{e^2k^2}{6\,\pi c^3}\dot{f}_z + \frac{e^2ak^3}{9\,\pi c^4}\ddot{f}_z - \frac{e^2a^2k^4}{18\,\pi c^5}\dddot{f}_z \cdots. \tag{29}$$

These equations show that it is necessary to distinguish between the coefficients of the longitudinal and transverse components of the acceleration and of each of its derivatives. All the coefficients approach infinity as the velocity of the electron approaches that of light, and the series cease to converge.

Including all terms not higher than the third order, the dynamical equation of an electron may be put in the form

$$\mathbf{K} = m\mathbf{f} - n\dot{\mathbf{f}} \cdots, \tag{30}$$

where
$$\mathbf{K} \equiv e\left\{\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H})\right\}$$

is the impressed force, and

$$m \equiv \frac{e^2}{6\,\pi a c^2},$$

$$n \equiv \frac{e^2}{6\,\pi c^3}.$$

22. Rigid body. Consider an element of volume $d\tau$ of a material body large enough to contain a vast number of positive and negative electrons, but small compared to the total volume of the body. A *rigid body* will be defined as one all such elements of which maintain the same relative configuration and, on the average, the same internal constitution with respect to the system in which the body is momentarily at rest, whatever external conditions the body may be subject to.

Consider a rigid body momentarily at rest in system S. Equations (27), (28), (29) give for the dynamical equation of an electron in this body

$$e\left\{\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H})\right\} = \frac{e^2}{6\,\pi a c^2}\mathbf{f} - \frac{e^2}{6\,\pi c^3}\dot{\mathbf{f}} \cdots, \tag{31}$$

through the third order, which is as far as the analysis will be carried in this section.

The electric and magnetic intensities appearing in this expression may be separated into the intensities $\mathbf{E}_e$ and $\mathbf{H}_e$ due to the impressed field, and the intensities $\mathbf{E}_0$ and $\mathbf{H}_0$ due to the fields

of the other electrons in the rigid body. For the latter, equations (13) and (14), section 16, give

$$\mathbf{E}_o = -\sum \frac{ek}{4\pi r^2}\left\{\mathbf{m}\left(1-\frac{3}{2}a^2+\frac{1}{2}b\right)+\frac{1}{2}\boldsymbol{\gamma}-\frac{2}{3}\boldsymbol{\delta}\cdots\right\},$$

$$\mathbf{H}_o = -\sum \frac{ek}{4\pi r^2}\{\boldsymbol{\beta}\times\mathbf{m}\cdots\},$$

where the summation extends over all the electrons except the one under consideration.

Suppose now that there is no external field and that the rigid body is permanently at rest in S. Symmetry requires that as many electrons in any element of volume $d\tau$ shall have a given velocity or acceleration in one direction due to the internal motion as in any other direction. This same condition must be satisfied in the presence of an impressed field, for the internal constitution of a rigid body is independent, by definition, of the external conditions to which it may be subject.

To return to the case of a rigid body momentarily at rest in S in the presence of an impressed field, let $\mathbf{f}_e$ and $\dot{\mathbf{f}}_e$ be the acceleration and rate of change of acceleration of the body as a whole, and $\mathbf{v}_o$, $\mathbf{f}_o$, and $\dot{\mathbf{f}}_o$ the velocity, acceleration, and rate of change of acceleration of an electron of the body due to the internal motion. Summing up over all the electrons,

$$\sum e\mathbf{E}_o = -\frac{1}{6\pi c^2}\left(\sum_{ij}\frac{e_i e_j}{r_{ij}}\right)\mathbf{f}_e + \frac{1}{6\pi c^3}\left(\sum_{ij} e_i e_j\right)\dot{\mathbf{f}}_e\cdots,$$

$$\frac{1}{c}\sum e\mathbf{v}_o\times\mathbf{H} = 0,$$

and therefore

$$\left(\sum_i e\right)\mathbf{E}_e = \left\{\frac{1}{6\pi c^2}\sum_i\frac{e_i^2}{a_i}+\frac{1}{6\pi c^2}\sum_{ij}\frac{e_i e_j}{r_{ij}}\right\}\mathbf{f}_e$$
$$-\left\{\frac{1}{6\pi c^3}\sum_i e_i^2+\frac{1}{6\pi c^3}\sum_{ij}e_i e_j\right\}\dot{\mathbf{f}}_e\cdots, \qquad (32)$$

where the double summation is for all values of both i and j such that

$$i \neq j.$$

Consider a rigid body at rest in S'. Then, dropping the subscript e,

$$\begin{aligned}(\sum_i e)\mathbf{E}' = &\left\{\frac{1}{6\pi c^2}\sum_i \frac{e_i^2}{a_i} + \frac{1}{6\pi c^2}\sum_{ij}\frac{e_i e_j}{r_{ij}}\right\}\mathbf{f}' \\ &- \left\{\frac{1}{6\pi c^3}\sum_i e_i^2 + \frac{1}{6\pi c^3}\sum_{ij} e_i e_j\right\}\dot{\mathbf{f}}' \ldots\end{aligned}$$

Substituting for $\mathbf{E}'$ its value in terms of $\mathbf{E}$ and $\mathbf{H}$, and for $\mathbf{f}'$ and $\dot{\mathbf{f}}'$ their values in terms of $\mathbf{f}$ and $\dot{\mathbf{f}}$, the dynamical equation of a rigid body takes the form

$$\begin{aligned}(\sum_i e)\left\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right\} = &\left\{\frac{1}{6\pi c^2}\sum_i \frac{e_i^2}{a_i} + \frac{1}{6\pi c^2}\sum_{ij}\frac{e_i e_j}{r_{ij}}\right\}\mathbf{f} \\ &- \left\{\frac{1}{6\pi c^3}\sum_i e_i^2 + \frac{1}{6\pi c^3}\sum_{ij} e_i e_j\right\}\dot{\mathbf{f}} \ldots \qquad (33)\end{aligned}$$

The first term in the brace multiplying $\mathbf{f}$ is the sum of the masses of the electrons composing the rigid body, while the second term is the sum of the *mutual masses* of these electrons. While the mass of an electron must be positive, the mutual mass of two may be positive or negative according as they have like or unlike signs. Hence the mass of a rigid body is greater the more electrons there are of the same sign. The same is true of the coefficient of the rate of change of acceleration. In fact this coefficient vanishes if the body is uncharged.

Denoting by $\mathbf{K}$ the impressed force, and by m and n the constant coefficients of $\mathbf{f}$ and $\dot{\mathbf{f}}$ respectively, the dynamical equation of a rigid body takes the form

$$\mathbf{K} = m\mathbf{f} - n\dot{\mathbf{f}} \ldots. \qquad (34)$$

A conductor carrying a current may be considered, in so far as the expressions arrived at in this section are concerned, as one rigid body through which another is passing. As the electrons carrying the current are all of the same sign, their mutual masses are positive, and the mass of the current is greater than the sum of the masses of the individual electrons which constitute it.

23. Experimental determination of charge and mass of electron. Consider a stream of negative electrons from the cathode of a discharge tube travelling at right angles to the lines of force of

mutually perpendicular electric and magnetic fields. The electromagnetic force on each electron is given by

$$\mathbf{K} = e\left\{\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right\}.$$

Hence if $$\frac{1}{c}\mathbf{v} \times \mathbf{H} = -\mathbf{E},$$

this force will vanish, and

$$v = \frac{E}{H}c.$$

By adjusting crossed electric and magnetic fields so as to produce no deflection in a beam of cathode rays, J. J. Thomson has found the velocity of these charged particles to be about one-tenth that of light.

If, now, the magnetic field be suppressed,

$$\mathbf{K} = e\mathbf{E},$$

and to a first approximation

$$m\mathbf{f} = e\mathbf{E}.$$

If the rays suffer a deflection d in travelling a distance s through this field, $$f = \frac{2\,dv^2}{s^2},$$

and the ratio of charge to mass is given in terms of measurable quantities by the expression

$$\frac{e}{m} = \frac{2\,dv^2}{Es^2}. \tag{35}$$

Similarly, if the electric field be suppressed, the value of the ratio of charge to mass may be obtained from the deflection suffered in traversing a magnetic field. In this case

$$\frac{e}{m} = \frac{2\,dvc}{Hs^2}. \tag{36}$$

By these methods it is found that

$$\frac{e}{m} = 1.77\,(10)^7 c\sqrt{4\,\pi} \tag{37}$$

for the negative electron.

Determinations of this ratio for beta rays moving with velocities only slightly less than the velocity of light verify the theoretical expression (20) for the increase of transverse mass with velocity.

In order to measure e and hence m other experimental methods are necessary. Suppose an electron to be attached to a minute oil drop situated between the horizontal plates of a parallel plate condenser. If the electric field is adjusted so that the oil drop remains at rest, its weight w is balanced by the force eE. Now if the drop is allowed to fall freely through the surrounding gas, its radius may be calculated from its rate of fall by Stokes' law. From the radius and density of the oil w may be determined and hence e computed. In this way the electronic charge has been found by Millikan to be

$$e = 4.77\,(10)^{-10}\sqrt{4\pi}, \tag{38}$$

and, combining this with (37), the mass of the negative electron is found to be

$$m = 9.0\,(10)^{-28}\,\text{gm.}, \tag{39}$$

which is about one eighteen-hundredth of the mass of a hydrogen atom. Hence the radius of the negative electron is

$$a = 1.88\,(10)^{-13}\,\text{cm.}$$

Since the positive electron has not yet been isolated, it has been impossible to measure its mass and radius, although there are reasons for supposing that it has a much greater mass and consequently a much smaller radius than the negative electron.

CHAPTER V

EQUATIONS OF THE ELECTROMAGNETIC FIELD

24. Divergence equations. The field due to a point charge is completely specified by any two of the three vector functions of position in space and time, **E**, **H**, and **c**. In the case of a complex field, **E** and **H** are the resultants of the electric and magnetic intensities respectively of the component simple fields, but **c** must be given for each elementary field. Hence in order to avoid explicit reference to the components of a complex field, as well as in order to give the field equations as great a symmetry as possible, the field is usually described in terms of **E** and **H**.

To find the divergence of the electric intensity due to any given distribution of point charges consider a small region $d\tau$ surrounded by the closed surface σ. By Gauss' theorem,

$$\nabla \cdot \mathbf{E}\, d\tau = \int \mathbf{E} \cdot d\boldsymbol{\sigma}.$$

Let letters with strokes over them refer to component fields.

Then $$\overline{\mathbf{E}}_{d\sigma} \equiv \frac{d\overline{N}}{d\sigma^2} d\boldsymbol{\sigma}$$

for each element of charge.

Therefore $$\nabla \cdot \mathbf{E}\, d\tau = \sum \int d\overline{N}.$$

Now the part of this sum due to charges outside the region $d\tau$ vanishes, while the part due to the charge de inside this region becomes equal to de itself. Hence

$$\nabla \cdot \mathbf{E} = \rho, \tag{1}$$

where ρ is the density of charge at the point in question.

In section 13 it was shown that

$$\mathbf{H} = \nabla \times \mathbf{a}.$$

Therefore $$\nabla \cdot \mathbf{H} = 0 \tag{2}$$

identically, or the divergence of the magnetic intensity is everywhere zero.

25. Vector fields. Any vector function of position in space and time may be represented by moving lines such as have been employed to give a geometrical significance to the electric intensity. These lines will be continuous at all points where the divergence of the vector function vanishes, as is obvious from the discussion contained in the preceding section. At other points lines will either begin or end. Equation (1) shows that the resultant electric intensity may be represented by continuous lines at all points except those at which electricity is present, while equation (2) shows that the resultant magnetic intensity may be represented everywhere by continuous lines.

Let **V** be a vector function whose magnitude and direction are represented by lines all points of which are moving with velocities of the same magnitude c. If dN tubes of these lines (a tube being a bundle of M lines) pass through a small surface ds_x with normal parallel to the X axis,

$$V_x \equiv \frac{dN}{ds_x}.$$

For the moment assume that no new lines are formed. Then in a time dt, V_x may suffer a change due to three causes. In the first place, the number of lines passing through ds_x may increase by virtue of the fact that the lines whose motion will bring them through this surface at the end of the time dt are more closely packed than those passing through ds_x originally. The increase in V_x due to this cause is

$$-\left\{\frac{\partial V_x}{\partial x} c_x + \frac{\partial V_x}{\partial y} c_y + \frac{\partial V_x}{\partial z} c_z\right\} dt.$$

Secondly, the velocity associated with the new lines may have a different direction from that of the old. This will produce a

crowding of the lines during the time dt and account for a change in V_x equal to

$$-\left\{V_x\frac{\partial c_y}{\partial y}+V_x\frac{\partial c_z}{\partial z}\right\}dt.$$

Finally, if $\mathbf{c}$ differs in direction at neighboring points on the same line, there will ensue a twisting of the lines which will produce in V_x an increase

$$\left\{V_y\frac{\partial c_x}{\partial y}+V_z\frac{\partial c_x}{\partial z}\right\}dt$$

in the time dt.

Therefore the total rate of change of V_x is

$$\frac{\partial V_x}{\partial t}=-\mathbf{c}\cdot\nabla V_x-V_x\nabla\cdot\mathbf{c}+\mathbf{V}\cdot\nabla c_x,$$

or

$$\frac{\partial\mathbf{V}}{\partial t}+\mathbf{c}\nabla\cdot\mathbf{V}=\nabla\times\{\mathbf{c}\times\mathbf{V}\}. \tag{3}$$

Now consider the increase in $\mathbf{V}$ due to the formation of new lines. Attention will be confined to those fields whose lines terminate only on sources. Let the points O etc. in Fig. 9 be each the source of a new line emitted in the direction of the arrows, the line sources themselves having a velocity $\mathbf{v}$ to the right. The number of tube sources per unit volume is obviously

$$\nabla\cdot\mathbf{V},$$

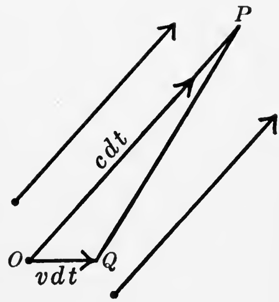

FIG. 9

and as the direction of the lines at P is QP, the increase in the value of $\mathbf{V}$ at P in a time dt due to the formation of new lines is

$$(\mathbf{c}-\mathbf{v})\nabla\cdot\mathbf{V}dt.$$

Therefore the complete expression for the rate of change of $\mathbf{V}$ becomes

$$\frac{\partial\mathbf{V}}{\partial t}+\mathbf{v}\nabla\cdot\mathbf{V}=\nabla\times\{\mathbf{c}\times\mathbf{V}\}. \tag{4}$$

If the field under consideration is due to the superposition of a number of simple fields of the character of those just discussed,

$$\frac{\partial \mathbf{V}}{\partial t}+\sum \overline{\mathbf{v}}\nabla\cdot\overline{\mathbf{V}}=\nabla\times\sum\{\overline{\mathbf{c}}\times\overline{\mathbf{V}}\}, \tag{5}$$

where the stroked letters refer to the component fields. Moreover, since a number of point sources emitting lines in different directions may be considered to coalesce, this equation applies equally well to point sources from which lines diverge in all directions. It is to be noted that equation (5) is a consequence of the properties of three-dimensional space, and nothing more.

26. Curl equations. Replace **V** by **E** in (5). Substituting from (1), the second term of the left-hand member becomes

$$\sum \overline{\rho}\overline{\mathbf{v}}.$$

Since, however, only one element of charge may occupy one point in space at a given time, the summation sign may be dropped.

The right-hand side of (5) becomes

$$\nabla\times\sum\{\overline{\mathbf{c}}_E\times\overline{\mathbf{E}}\},$$

where $\overline{\mathbf{c}}_E$ is the velocity of a moving element of an electric line of force of one of the component fields.

But
$$\mathbf{H}\equiv\frac{1}{c}\sum\overline{\mathbf{c}}_E\times\overline{\mathbf{E}}.$$

Hence
$$\nabla\times\mathbf{H}=\frac{1}{c}\{\dot{\mathbf{E}}+\rho\mathbf{v}\}. \tag{6}$$

If **V** is replaced by **H** in (5), it follows from (2) that the second term of the left-hand member vanishes. Hence the equation becomes

$$\nabla\times\sum\frac{1}{c}\{\overline{\mathbf{c}}_H\times\overline{\mathbf{H}}\}=\frac{1}{c}\dot{\mathbf{H}}, \tag{7}$$

where the velocity $\overline{\mathbf{c}}_H$ of a moving element of a magnetic line of force of one of the component fields does not in general have the same direction as the velocity $\overline{\mathbf{c}}_E$ of a moving element of an electric line of force of the same field.

But if $\mathbf{a}$ is eliminated from the equations obtained by curling (20) and differentiating (21) with respect to the time, in section 13, it is found that

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}. \tag{8}$$

Hence, as (7) and (8) apply equally well to simple or complex fields, comparison of these two equations shows that for a single elementary field

$$\overline{\mathbf{E}} = -\frac{1}{c}\bar{\mathbf{c}}_H \times \overline{\mathbf{H}} + \nabla\overline{\psi}, \tag{9}$$

a relation which is complementary to the definition

$$\overline{\mathbf{H}} \equiv \frac{1}{c}\bar{\mathbf{c}}_E \times \overline{\mathbf{E}}. \tag{10}$$

Therefore

$$\overline{\mathbf{E}} = -\frac{1}{c^2}\bar{\mathbf{c}}_H \times (\bar{\mathbf{c}}_E \times \overline{\mathbf{E}}) + \nabla\overline{\psi},$$

or

$$\overline{\mathbf{E}}\left(1 - \frac{\bar{\mathbf{c}}_E \cdot \bar{\mathbf{c}}_H}{c^2}\right) = -\frac{1}{c^2}\bar{\mathbf{c}}_H \cdot \nabla\overline{\psi}\,\bar{\mathbf{c}}_E + \nabla\overline{\psi};$$

and

$$\overline{\mathbf{H}} = -\frac{1}{c^2}\bar{\mathbf{c}}_E \times (\bar{\mathbf{c}}_H \times \overline{\mathbf{H}}) + \frac{1}{c}\bar{\mathbf{c}}_E \times \nabla\overline{\psi},$$

or

$$\overline{\mathbf{H}}\left(1 - \frac{\bar{\mathbf{c}}_E \cdot \bar{\mathbf{c}}_H}{c^2}\right) = \frac{1}{c}\bar{\mathbf{c}}_E \times \nabla\overline{\psi}.$$

Hence it follows that $\bar{\mathbf{c}}_H$ is parallel to $\bar{\mathbf{c}}_E$ when, and only when, $\nabla\overline{\psi}$ is parallel to $\bar{\mathbf{c}}_E$.

27. Electrodynamic equations. Equations (1), (2), (6), (8) specify the electromagnetic field in terms of the distribution of density of charge ρ and velocity $\mathbf{v}$. If (2), section 18, is written as an equation instead of as a definition in order to signify that it includes the dynamical assumption immediately following it, this equation suffices to determine the effect of an electromagnetic field upon matter. These five equations contain the whole of electrodynamics. Collected they are:

$$\nabla \cdot \mathbf{E} = \rho, \tag{11}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{13}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}, \tag{12}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\{\dot{\mathbf{E}} + \rho\mathbf{v}\}, \tag{14}$$

$$\mathbf{F} = \rho\left\{\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H})\right\}, \tag{15}$$

where $\mathbf{F}$ is the electromagnetic force per unit volume.

28. Energy relations. From (12) and (14) it follows that

$$\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H}=-\frac{1}{c}(\mathbf{E}\cdot\dot{\mathbf{E}}+\mathbf{H}\cdot\dot{\mathbf{H}})-\frac{1}{c}\rho\mathbf{E}\cdot\mathbf{v}.$$

But
$$\mathbf{H}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H}\equiv\nabla\cdot(\mathbf{E}\times\mathbf{H}),$$
and
$$\rho\mathbf{E}\cdot\mathbf{v}=\mathbf{F}\cdot\mathbf{v}.$$

Hence
$$\frac{d}{dt}\left\{\frac{1}{2}(E^2+H^2)\right\}+c\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{F}\cdot\mathbf{v}=0.$$

Integrating over any arbitrarily chosen portion of space, and applying Gauss' theorem to the second term,

$$\frac{d}{dt}\int\frac{1}{2}(E^2+H^2)\,d\tau+c\int(\mathbf{E}\times\mathbf{H})\cdot d\boldsymbol{\sigma}+\int\mathbf{F}\cdot\mathbf{v}\,d\tau=0, \quad (16)$$

where $d\tau$ is an element of volume, and $d\boldsymbol{\sigma}$ an element of the bounding surface having the direction of the outward drawn normal. The third term of this expression measures the rate at which work is done *by* the electromagnetic field on the matter contained in the region selected. Hence, if the principle of conservation of energy is to hold, the first two terms must be interpreted as the rate at which the energy of the field increases plus the rate at which energy escapes through the surface bounding the field.

Suppose **E** and **H** to be zero everywhere on this surface. Then no energy escapes from the region enclosed, and the rate at which work is done *on* the electromagnetic field equals the rate at which its energy increases. But the integrand of the second term of (16) is everywhere zero. Hence the rate of increase of energy of the field must be represented by the first term, the form of which suggests that

$$u=\tfrac{1}{2}(E^2+H^2)$$

is to be considered as the electromagnetic energy per unit volume.

The second term of (16), then, must be interpreted as the outward flux of energy through the surface enveloping the field, and its form suggests that

$$\mathbf{s}=c(\mathbf{E}\times\mathbf{H})$$

is to be considered as the flow of energy per unit cross section per unit time.

29. Electromagnetic waves in space. For empty space the electromagnetic equations take the form

$$\nabla\cdot\mathbf{E}=0, \qquad (17) \qquad\qquad \nabla\cdot\mathbf{H}=0, \qquad (19)$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\dot{\mathbf{H}}, \qquad (18) \qquad\qquad \nabla\times\mathbf{H}=\frac{1}{c}\dot{\mathbf{E}}. \qquad (20)$$

To eliminate $\mathbf{H}$, curl (18) and differentiate (20) with respect to the time. Thus

$$\nabla\times\nabla\times\mathbf{E}=-\frac{1}{c}\nabla\times\dot{\mathbf{H}}$$

$$=-\frac{1}{c^2}\ddot{\mathbf{E}}.$$

But
$$\nabla\times\nabla\times\mathbf{E}\equiv\nabla\nabla\cdot\mathbf{E}-\nabla\cdot\nabla\mathbf{E}$$

$$=-\nabla\cdot\nabla\mathbf{E}.$$

Therefore
$$\nabla\cdot\nabla\mathbf{E}-\frac{1}{c^2}\ddot{\mathbf{E}}=0, \qquad (21)$$

and, similarly,
$$\nabla\cdot\nabla\mathbf{H}-\frac{1}{c^2}\ddot{\mathbf{H}}=0. \qquad (22)$$

These are equations of waves moving with velocity c.

Consider a plane wave advancing along the X axis. Then $\mathbf{E}$ is a function of x and t only, and (21) reduces to

$$\frac{\partial^2\mathbf{E}}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0. \qquad (23)$$

The solution of this equation for the case of a wave moving in the positive X direction is

$$E_x=f(x-ct),$$

$$E_y=g(x-ct),$$

$$E_z=h(x-ct).$$

Hence it follows from (17) that E_x is a constant, and since the present discussion is concerned only with the variable part of the field, this constant may be taken as zero.

Therefore
$$E_x = 0, \tag{24}$$
$$E_y = g(x - ct), \tag{25}$$
$$E_z = h(x - ct). \tag{26}$$

From (18) it follows that

$$\dot{H}_x = 0,$$
$$\dot{H}_y = ch'(x - ct),$$
$$\dot{H}_z = -cg'(x - ct),$$

and hence
$$H_x = 0, \tag{27}$$
$$H_y = -h(x - ct), \tag{28}$$
$$H_z = g(x - ct), \tag{29}$$

except for a possible constant of integration. Therefore the variable parts of E and H have the same magnitude at any point and time, and lie in a plane at right angles to the direction of propagation.

The cosine of the angle between **E** and **H** is proportional to

$$\begin{aligned} E_yH_y + E_zH_z &= -gh + gh \\ &= 0, \end{aligned}$$

showing that **E** and **H** are at right angles in the wave front.

The preceding section gave the flow of energy as

$$\begin{aligned} c(\mathbf{E} \times \mathbf{H}) &= c(g^2 + h^2)\mathbf{i} \\ &= c\left\{\frac{1}{2}(E^2 + H^2)\right\}\mathbf{i}, \end{aligned} \tag{30}$$

showing that the propagation of energy is along the X axis and that the entire energy of the wave front is advancing with the velocity of light. It follows from this equation that the direction of propagation of the wave is at right angles to the plane of **E** and **H** in the sense of

$$\mathbf{E} \times \mathbf{H}.$$

Consider a spherical wave having its center at the origin. In this case **E** is a function of the radius vector r and t only, and (21) reduces to

$$\frac{\partial^2 \mathbf{E}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathbf{E}}{\partial r} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{31}$$

of which the solution is

$$E_r = \frac{1}{r} f(r \pm ct),$$

$$E_\theta = \frac{1}{r} g(r \pm ct),$$

$$E_\phi = \frac{1}{r} h(r \pm ct).$$

As in the case of the plane wave, it may be shown that **E** and **H** are mutually perpendicular, and at right angles to the direction of propagation. If $\mathbf{r}_1$ is a unit vector along the outward drawn normal to the wave front, the flow of energy is

$$c(\mathbf{E} \times \mathbf{H}) = \pm c\frac{g^2 + h^2}{r^2}\mathbf{r}_1$$

$$= \pm c\left\{\frac{1}{2}(E^2 + H^2)\right\}\mathbf{r}_1, \tag{32}$$

showing that the amount of energy passing through unit cross section in unit time varies inversely with the square of the distance from the source, and that the entire energy of the wave front is advancing with the velocity of light.

30. Radiation pressure. Substituting in (15) the values of ρ and $\rho\mathbf{v}$ from (11) and (14), the electromagnetic force per unit volume takes the form

$$\mathbf{F} = \nabla\cdot\mathbf{E}\mathbf{E} + (\nabla \times \mathbf{H}) \times \mathbf{H} - \frac{1}{c}\dot{\mathbf{E}} \times \mathbf{H},$$

and making use of (12) and (13),

$$\mathbf{F} = -\frac{1}{c}\frac{d}{dt}(\mathbf{E} \times \mathbf{H}) + \nabla\cdot\mathbf{E}\mathbf{E} + \nabla\cdot\mathbf{H}\mathbf{H} + (\nabla \times \mathbf{E}) \times \mathbf{E} + (\nabla \times \mathbf{H}) \times \mathbf{H}.$$

The total electromagnetic force on the matter in any given region is

$$\mathbf{K} = \int \mathbf{F} d\tau$$

$$= \mathbf{K}_\tau + \mathbf{K}_\sigma,$$

where

$$\mathbf{K}_\tau \equiv -\frac{1}{c}\int \frac{d}{dt}(\mathbf{E} \times \mathbf{H})\, d\tau$$

$$= -\frac{1}{c^2}\frac{d}{dt}\int \mathbf{s}\, d\tau, \tag{33}$$

and

$$\mathbf{K}_\sigma \equiv \int \{\nabla\cdot\mathbf{E}\mathbf{E} + \nabla\cdot\mathbf{H}\mathbf{H} + (\nabla \times \mathbf{E}) \times \mathbf{E} + (\nabla \times \mathbf{H}) \times \mathbf{H}\}\, d\tau.$$

Now let $\mathbf{J} \equiv \int \{\nabla\cdot\mathbf{E}\mathbf{E} + (\nabla \times \mathbf{E}) \times \mathbf{E}\}\, d\tau$

$$= \int \{\nabla\cdot\mathbf{E}\mathbf{E} + \mathbf{E}\cdot\nabla\mathbf{E} - \tfrac{1}{2}\nabla(\mathbf{E}\cdot\mathbf{E})\}\, d\tau$$

$$= \int \{\nabla\cdot(\mathbf{E}\mathbf{E}) - \tfrac{1}{2}\nabla(\mathbf{E}\cdot\mathbf{E})\}\, d\tau.$$

By Gauss' theorem,

$$\int \nabla\cdot(\mathbf{E}\mathbf{E})\, d\tau = \int \mathbf{E}\mathbf{E}\cdot d\boldsymbol{\sigma},$$

and

$$\int \nabla(\mathbf{E}\cdot\mathbf{E})\, d\tau = \mathbf{i}\int \nabla\cdot(\mathbf{i}\mathbf{E}\cdot\mathbf{E})\, d\tau + \text{etc.}$$

$$= \mathbf{i}\int \mathbf{E}\cdot\mathbf{E}\mathbf{i}\cdot d\boldsymbol{\sigma} + \text{etc.}$$

$$= \int \mathbf{E}\cdot\mathbf{E}\, d\boldsymbol{\sigma}.$$

Therefore $\mathbf{J} = \int \mathbf{E}\mathbf{E}\cdot d\boldsymbol{\sigma} - \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{E}\, d\boldsymbol{\sigma},$

and $\mathbf{K}_\sigma$ is given entirely by the surface integral

$$\mathbf{K}_\sigma = \int (\mathbf{E}\mathbf{E} + \mathbf{H}\mathbf{H})\cdot d\boldsymbol{\sigma} - \tfrac{1}{2}\int (\mathbf{E}\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{H})\, d\boldsymbol{\sigma} \tag{34}$$

taken over the surface bounding the chosen region.

Consider a region which is the seat of either stationary radiation or undamped periodic vibrations. In the former case the

average $\mathbf{K}_\tau$ is zero, and in the latter case $\mathbf{K}_\tau$ vanishes provided the value of the force desired is the average over a whole number of periods. Hence the force on the matter within the region is given entirely by $\mathbf{K}_\sigma$.

Consider a box (Fig. 10) with perfectly *reflecting* walls containing homogeneous isotropic radiation. Describe the pill-box shaped surface $ABCD$ about an element $d\sigma$ of that part of the wall of the box which is perpendicular to the X axis. The only

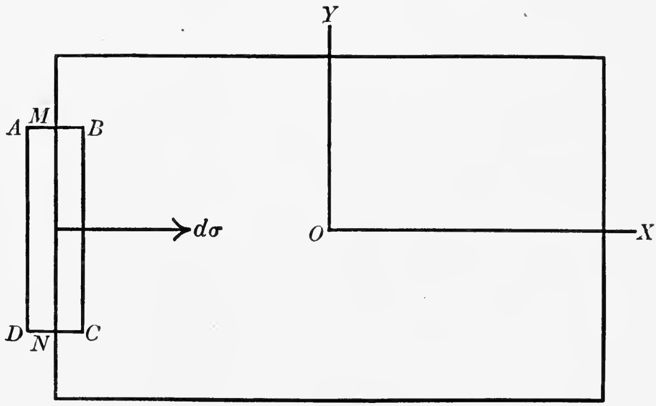

FIG. 10

matter within this closed surface is the portion MN of the wall of the box. In so far as the effect of the radiation inside the box is concerned, the average force on MN is given entirely by $\mathbf{K}_\sigma$, which reduces to an integral over BC, since AD lies completely outside the field, and the sides AB, CD of the pill-box have a negligible area. Hence if X_x, Y_x, Z_x are the forces per unit area on the surface MN of the box due to the pressure of the radiation inside it, parallel respectively to the X, Y, Z axes,

$$X_x = \tfrac{1}{2}(E_x^2 - E_y^2 - E_z^2) + \tfrac{1}{2}(H_x^2 - H_y^2 - H_z^2), \qquad (35)$$

$$Y_x = E_x E_y + H_x H_y, \qquad (36)$$

$$Z_x = E_x E_z + H_x H_z, \qquad (37)$$

and similar expressions hold for the parts of the wall perpendicular to the Y and Z axes.

As the radiation is assumed homogeneous and isotropic,

$$\overline{E}_x = \overline{E}_y = \overline{E}_z = 0,$$

$$\overline{E_x^2} = \overline{E_y^2} = \overline{E_z^2} = \tfrac{1}{3}\overline{E^2},$$

where the stroke denotes the average value of the quantity over which it is placed. Similar relations hold for the components of the magnetic intensity. Therefore

$$X_x = -\tfrac{1}{6}(\overline{E^2} + \overline{H^2})$$
$$= -\tfrac{1}{3}u, \qquad (38)$$
$$Y_x = Z_x = 0, \qquad (39)$$

showing that the radiation within the box exerts a normal stress on the walls equal to one third of the electromagnetic energy per unit volume.

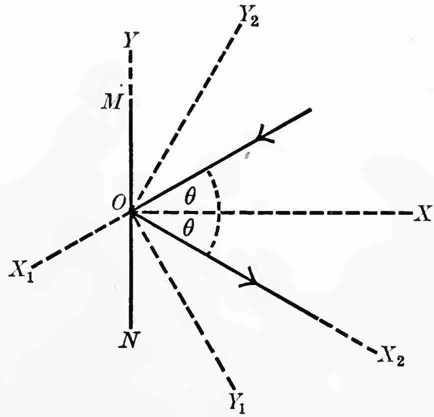

FIG. 11

Consider, now, a box similar in shape to that of Fig. 10 with perfectly *conducting* walls. Instead of being filled with homogeneous radiation, suppose the box to contain a train of undamped plane waves which are incident on the surface MN (Fig. 11) at the angle θ.

Referred to the $X_1Y_1Z_1$ axes in the figure, equations (24) to (29) give for the incident wave

$$E_{x_1} = 0,$$
$$E_{y_1} = g_1(x - ct),$$
$$E_{z_1} = h_1(x - ct),$$

and

$$H_{x_1} = 0,$$
$$H_{y_1} = -h_1(x - ct),$$
$$H_{z_1} = g_1(x - ct).$$

Therefore the components of the electric and magnetic intensities along the XYZ axes due to the incident wave alone, are

$$
\begin{aligned}
E_x &= g_1 \sin\theta, \\
E_y &= -g_1 \cos\theta, \\
E_z &= h_1, \\
H_x &= -h_1 \sin\theta, \\
H_y &= h_1 \cos\theta, \\
H_z &= g_1,
\end{aligned}
$$

where the argument of the arbitrary g_1 and h_1 functions is

$$x_1 - ct = -x\cos\theta - y\sin\theta - ct.$$

Similarly, for the reflected wave,

$$
\begin{aligned}
E_x &= g_2 \sin\theta, \\
E_y &= g_2 \cos\theta, \\
E_z &= h_2, \\
H_x &= -h_2 \sin\theta, \\
H_y &= -h_2 \cos\theta, \\
H_z &= g_2,
\end{aligned}
$$

where the argument of the g_2 and h_2 functions is

$$x_2 - ct = x\cos\theta - y\sin\theta - ct.$$

At the reflecting surface the arguments of the functions with different subscripts become the same. Moreover, as this surface is perfectly conducting, the tangential component of the resultant electric intensity must vanish. Therefore

$$
\begin{aligned}
-g_1\cos\theta + g_2\cos\theta &= 0, \\
h_1 + h_2 &= 0,
\end{aligned}
$$

or

$$
\begin{aligned}
g_1 &= g_2 \equiv g, \\
h_1 &= -h_2 \equiv h.
\end{aligned}
$$

Hence it follows from (30) that all the energy brought up in the incident wave is carried away in the reflected wave.

Therefore *a perfectly conducting surface is a perfect reflector of electromagnetic waves.*

The resultant electric and magnetic intensities just to the right of MN are

$$E_x = 2g\sin\theta, \tag{40}$$

$$E_y = 0, \tag{41}$$

$$E_z = 0, \tag{42}$$

$$H_x = 0, \tag{43}$$

$$H_y = 2h\cos\theta, \tag{44}$$

$$H_z = 2g. \tag{45}$$

Hence, as there is no field to the left of MN, this surface must have a charge $2g\sin\theta$ per unit area. If g is a simple harmonic function, this charge will be alternately positive and negative.

Substituting the values (40) to (45) in the expressions (35), (36), (37) for the components of the stress on the reflecting surface,

$$X_x = -2(g^2 + h^2)\cos^2\theta, \tag{46}$$

$$Y_x = Z_x = 0, \tag{47}$$

showing that the stress due to the radiation is a pressure normal to the surface.

Now, the energy density of the incident wave is

$$u_1 = g^2 + h^2,$$

and therefore

$$X_x = -2u_1\cos^2\theta. \tag{48}$$

Let I stand for the energy striking the reflecting surface per unit area per unit time. Then

$$I = u_1 c\cos\theta,$$

and

$$X_x = -2\frac{I}{c}\cos\theta. \tag{49}$$

Suppose that instead of containing a single train of plane waves, the box is filled with plane waves travelling in all directions at random; that is, with a homogeneous isotropic radiation. Then if u is the energy density of the radiation, the energy per

unit volume of that portion of the radiation which is incident on MN between the angles θ and $\theta + d\theta$ is

$$u_1 = \frac{1}{2} u \sin\theta d\theta,$$

and

$$X_x = -2\sum u_1 \cos^2\theta$$

$$= -u \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta d\theta$$

$$= -\frac{1}{3} u,$$

agreeing with (38).

31. Electromagnetic momentum. Consider a closed surface $ABCD$ (Fig. 12), surrounding some matter. Let the field inside this surface be in a stationary state. Then the force on the matter contained is given by $\mathbf{K}_\sigma$ integrated over the bounding surface, the outward-drawn normal being positive. Denote by $\mathbf{K}_{\sigma 1}$ the part of $\mathbf{K}_\sigma$ obtained by integrating over BC. Then the force $\mathbf{K}$ on the matter under consideration, due to the electromagnetic field extending through the surface BC, is equal to $\mathbf{K}_{\sigma_1}$.

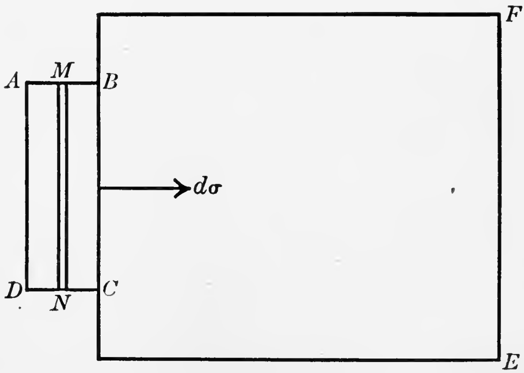

Fig. 12

Let $BCEF$ be a closed surface which surrounds no matter, such that the radiation field inside it comes into contact with the surface only between B and C. Then the value of $\mathbf{K}_\sigma$ integrated over this surface is equal to $-\mathbf{K}_{\sigma_1}$, since the outward-drawn normal to BC in this case has the opposite direction to that for $ABCD$. Hence

$$-\mathbf{K}_{\sigma_1} + \mathbf{K}_\tau = 0,$$

as no matter is present inside the surface. Therefore

$$\mathbf{K} - \mathbf{K}_\tau = 0, \tag{50}$$

where $\mathbf{K}$ is the force exerted by the electromagnetic field in $BCEF$ on the matter in $ABCD$. Hence, if the law of action and reaction is to hold, $-\mathbf{K}_\tau$ must be interpreted as the force exerted by the matter in $ABCD$ on the electromagnetic field in $BCEF$. The form of this expression, namely,

$$-\mathbf{K}_\tau = \frac{d}{dt}\left\{\frac{1}{c^2}\int \mathbf{s}\,d\tau\right\},$$

suggests that

$$\frac{1}{c^2}\mathbf{s}$$

is to be considered as the momentum per unit volume of the electromagnetic field.

Suppose the matter in $ABCD$ to consist of a rigid body with a plane, perfectly reflecting surface just to the left of BC. Let a limited train of plane waves be incident on this surface at the angle θ. Reference to (30) shows that the electromagnetic momentum per unit volume of the incident radiation is

$$\frac{1}{c^2}s = \frac{u_1}{c}. \tag{51}$$

Therefore the momentum of the radiation striking each unit area of the surface in unit time is $u_1 \cos\theta$, and the momentum of the reflected radiation is of the same magnitude. The force exerted by each unit area of the reflecting surface on the train of waves is equal to the vector increase in momentum per unit time; that is,

$$-K_\tau = 2\,u_1 \cos^2\theta$$

along the outward-drawn normal. Consequently the stress exerted by the radiation on this surface is

$$\begin{aligned} K &= -\,2\,u_1 \cos^2\theta \\ &= -\,2\frac{I}{c}\cos\theta, \end{aligned}$$

agreeing with (48) and (49).

32. Four-dimensional representation. In four-dimensional space two mutually perpendicular lines may be drawn at right angles to an element of surface. Consequently the vector properties of such an element cannot be expressed by the direction of its normal. In fact, it is necessary to distinguish between directed

linear elements, directed surface elements, and directed volume elements. The first have as components their projections on the four coördinate axes, the second their projections on the six coördinate planes, and the third their projections on the four coördinate planoids. Hence the first and third are often called four-vectors, and the second six-vectors. Here they will be called vectors of the first, second, and third orders respectively.

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three vectors of the first order. Then the vector product $\mathbf{a} \times \mathbf{b}$ is a directed surface having for its area the parallelogram of which $\mathbf{a}$ and $\mathbf{b}$ are the sides, and so directed that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

Similarly, $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ is a directed volume having for its magnitude that of the parallelepiped whose edges are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. In the case of any cross product, interchange of two adjacent vectors changes the sign of the product.

Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ be unit vectors of the first order parallel respectively to the X, Y, Z, L axes. Then

$$\mathbf{k}_{23} \equiv \mathbf{k}_2 \times \mathbf{k}_3$$

is a unit vector of the second order in the YZ coördinate plane, and

$$\begin{aligned} \mathbf{k}_{341} &\equiv \mathbf{k}_3 \times \mathbf{k}_4 \times \mathbf{k}_1 \\ &= \mathbf{k}_{34} \times \mathbf{k}_1 \end{aligned}$$

is a unit vector of the third order in the ZLX coördinate planoid. From their definition it is obvious that two adjacent subscripts of a unit vector of the second or third order may be interchanged provided the sign of the vector is changed.

The dot product of two unit vectors is defined as follows. If the vector of lower order has a subscript which the other lacks, the dot product vanishes. Otherwise, the product is the unit vector remaining after like digits in the subscript of each vector have been brought to the end and cancelled. Thus

$$\begin{aligned} \mathbf{k}_{124} \cdot \mathbf{k}_{12} &= \mathbf{k}_{412} \cdot \mathbf{k}_{12} \\ &= \mathbf{k}_4 ; \\ \mathbf{k}_{13} \cdot \mathbf{k}_{13} &= 1 ; \\ \mathbf{k}_{14} \cdot \mathbf{k}_{13} &= 0. \end{aligned}$$

Thus, while the order of the cross product of two unit vectors is equal to the sum of their orders, the order of the dot product is equal to the difference of the orders of the two factors.

Consider the vector of the second order

$$\mathbf{M} \equiv M_{xy}\mathbf{k}_{12} + M_{yz}\mathbf{k}_{23} + M_{zx}\mathbf{k}_{31} + M_{xl}\mathbf{k}_{14} + M_{yl}\mathbf{k}_{24} + M_{zl}\mathbf{k}_{34}. \quad (52)$$

The dual $\mathbf{M}^{\times}$ of $\mathbf{M}$ is defined as a vector of the same order with components such that

$$M^{\times}_{mn} = M_{op},$$

where $mnop$ is formed from $xyzl$ by an even number of interchanges of adjacent letters. Hence

$$\mathbf{M}^{\times} \equiv M_{zl}\mathbf{k}_{12} + M_{xl}\mathbf{k}_{23} + M_{yl}\mathbf{k}_{31} + M_{yz}\mathbf{k}_{14} + M_{zx}\mathbf{k}_{24} + M_{xy}\mathbf{k}_{34}. \quad (53)$$

If

$$\mathbf{P} \equiv P_x\mathbf{k}_1 + P_y\mathbf{k}_2 + P_z\mathbf{k}_3 + P_l\mathbf{k}_4, \quad (54)$$

the rule for forming the dot product shows that

$$\begin{aligned} \mathbf{P}\cdot\mathbf{M} \equiv (\quad & \qquad\quad + P_yM_{xy} - P_zM_{zx} + P_lM_{xl})\mathbf{k}_1 \\ + (- P_xM_{xy} & \qquad\qquad\qquad + P_zM_{yz} + P_lM_{yl})\mathbf{k}_2 \\ + (P_xM_{zx} & - P_yM_{yz} \qquad\qquad\quad + P_lM_{zl})\mathbf{k}_3 \\ + (- P_xM_{xl} & - P_yM_{yl} - P_zM_{zl} \qquad\qquad\quad)\mathbf{k}_4. \end{aligned} \quad (55)$$

In four-dimensional analysis the vector operator

$$\Diamond \equiv \mathbf{k}_1\frac{\partial}{\partial x} + \mathbf{k}_2\frac{\partial}{\partial y} + \mathbf{k}_3\frac{\partial}{\partial z} + \mathbf{k}_4\frac{\partial}{\partial l}$$

plays much the same part as ∇ in three dimensions.

Consider the product

$$\begin{aligned} \Diamond\cdot\mathbf{M} \equiv & \left(\qquad\quad + \frac{\partial M_{xy}}{\partial y} - \frac{\partial M_{zx}}{\partial z} + \frac{\partial M_{xl}}{\partial l}\right)\mathbf{k}_1 \\ & + \left(- \frac{\partial M_{xy}}{\partial x} \qquad\quad + \frac{\partial M_{yz}}{\partial z} + \frac{\partial M_{yl}}{\partial l}\right)\mathbf{k}_2 \\ & + \left(\frac{\partial M_{zx}}{\partial x} - \frac{\partial M_{yz}}{\partial y} \qquad\quad + \frac{\partial M_{zl}}{\partial l}\right)\mathbf{k}_3 \\ & + \left(- \frac{\partial M_{xl}}{\partial x} - \frac{\partial M_{yl}}{\partial y} - \frac{\partial M_{zl}}{\partial z} \qquad\quad \right)\mathbf{k}_4. \end{aligned} \quad (56)$$

Remembering that $l \equiv ict$, equations (11) and (14) of the electromagnetic field may be written

$$\begin{aligned}
&+\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{\partial iE_x}{\partial l} = \frac{1}{c}\rho v_x,\\
-\frac{\partial H_z}{\partial x} \quad &+\frac{\partial H_x}{\partial z} - \frac{\partial iE_y}{\partial l} = \frac{1}{c}\rho v_y,\\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \quad &- \frac{\partial iE_z}{\partial l} = \frac{1}{c}\rho v_z,\\
\frac{\partial iE_x}{\partial x} + \frac{\partial iE_y}{\partial y} + \frac{\partial iE_z}{\partial z} \quad &= i\rho,
\end{aligned}$$

and equations (12) and (13)

$$\begin{aligned}
&-\frac{\partial iE_z}{\partial y} + \frac{\partial iE_y}{\partial z} + \frac{\partial H_x}{\partial l} = 0,\\
\frac{\partial iE_z}{\partial x} \quad &- \frac{\partial iE_x}{\partial z} + \frac{\partial H_y}{\partial l} = 0,\\
-\frac{\partial iE_y}{\partial x} + \frac{\partial iE_x}{\partial y} \quad &+ \frac{\partial H_z}{\partial l} = 0,\\
-\frac{\partial H_x}{\partial x} - \frac{\partial H_y}{\partial y} - \frac{\partial H_z}{\partial z} \quad &= 0.
\end{aligned}$$

Comparison with (52), (53), (54), and (56) shows that if

$$\mathbf{M} \equiv H_z\mathbf{k}_{12} + H_x\mathbf{k}_{23} + H_y\mathbf{k}_{31} - iE_x\mathbf{k}_{14} - iE_y\mathbf{k}_{24} - iE_z\mathbf{k}_{34},$$

and $$\mathbf{P} \equiv \frac{1}{c}\rho v_x\mathbf{k}_1 + \frac{1}{c}\rho v_y\mathbf{k}_2 + \frac{1}{c}\rho v_z\mathbf{k}_3 + i\rho\mathbf{k}_4,$$

the two scalar equations (11) and (13) and the two vector equations (12) and (14) of the electromagnetic field are expressed by the pair of four-dimensional vector equations

$$\Diamond\cdot\mathbf{M} = \mathbf{P}, \tag{57}$$

$$\Diamond\cdot\mathbf{M}^{\times} = 0. \tag{58}$$

The equation (15), which gives the effect of an electromagnetic field on matter, and the energy equation may be written together in the form

$$
\begin{aligned}
F_x &= \qquad\qquad\qquad + \frac{1}{c}\rho v_y H_z - \frac{1}{c}\rho v_z H_y + \rho E_x, \\
F_y &= -\frac{1}{c}\rho v_x H_z \qquad\qquad + \frac{1}{c}\rho v_z H_x + \rho E_y, \\
F_z &= \frac{1}{c}\rho v_x H_y - \frac{1}{c}\rho v_y H_x \qquad\qquad + \rho E_z, \\
\frac{i}{c}\frac{du}{dt} &= \frac{1}{c}\rho v_x i E_x + \frac{1}{c}\rho v_y i E_y + \frac{1}{c}\rho v_z i E_z.
\end{aligned}
$$

Hence, if $$F \equiv F_x \mathbf{k}_1 + F_y \mathbf{k}_2 + F_z \mathbf{k}_3 + \frac{i}{c}\frac{du}{dt}\mathbf{k}_4,$$

these relations are contained in the four-dimensional vector equation

$$\mathbf{F} = \mathbf{P}\cdot\mathbf{M}. \tag{59}$$

The scalar operator

$$\Diamond\cdot\Diamond \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial l^2}$$

is known as the d'Alembertian. In four-dimensional analysis the wave equations (21) and (22) are expressed by

$$\Diamond\cdot\Diamond\,\mathbf{M} = 0. \tag{60}$$

In addition to those just mentioned, many other electrodynamic relations may be expressed in far more compact form in terms of four-dimensional vector analysis than in the analysis of three dimensions.

CHAPTER VI

RADIATION

33. Radiation from a single electron. Let $\mathbf{v}$ and $\mathbf{f}$ be the electron's velocity and acceleration respectively at the time t. Take its position at this time as origin, and choose axes so that the X axis is parallel to $\mathbf{v}$ and the XY plane contains $\mathbf{f}$. Describe about the origin a sphere of radius r, large compared with the radius of the electron. Consider an element $d\sigma$ of the surface of this sphere such that the radius vector drawn to it from the origin makes an angle θ with the X axis. The energy emitted from the electron during an interval of time dt in the direction of this radius vector will reach the surface of the sphere at a time $t + r/c$, and will take a time

$$(1 - \beta \cos\theta)\, dt$$

to pass through this surface. Hence, as the flow of energy per unit cross section per unit time is equal to

$$\mathbf{s} = c\,(\mathbf{E} \times \mathbf{H}),$$

the energy emitted by the electron during the time dt is given by the integral

$$R dt = \int \mathbf{s} \cdot d\boldsymbol{\sigma}\,(1 - \beta \cos\theta)\, dt$$

taken over the surface of the sphere. Therefore the rate of radiation from the electron is

$$R = \int \mathbf{s} \cdot d\boldsymbol{\sigma}\,(1 - \beta \cos\theta). \qquad (1)$$

As $$\mathbf{H} \equiv \frac{1}{c}\, \mathbf{c} \times \mathbf{E},$$

it follows that $$\mathbf{s} = E^2 \mathbf{c} - \mathbf{E} \cdot \mathbf{c} \mathbf{E}.$$

The sphere over which the surface integral is to be taken may be made as large as desired. Consequently the terms in

equation (14), section 12, for the electric intensity which involve the inverse square of the radius vector may be made so small compared with the term involving the inverse first power of this quantity that they may be neglected. As the term involving the inverse first power defines a vector perpendicular to **c**,

$$\mathbf{s} = E^2\mathbf{c}$$

$$= \frac{e^2\mathbf{c}}{16\,\pi^2 r^2 c^4}\left\{\frac{f^2}{(1-\beta\cos\theta)^4} + \frac{2\,\beta f_r f_x}{(1-\beta\cos\theta)^5} - \frac{(1-\beta^2)f_r^2}{(1-\beta\cos\theta)^6}\right\}, \quad (2)$$

where f_r is the component of the electron's acceleration along the radius vector. For small values of β this becomes approximately

$$\mathbf{s} = \frac{e^2 f^2 \sin^2\alpha}{16\,\pi^2 r^2 c^4}\,\mathbf{c},$$

where α is the angle which the radius vector makes with **f**. Hence the radiation vanishes in the direction of **f** and is maximum at right angles to this direction.

To find the rate of total radiation from the electron, integral (1) must be evaluated. Substituting the expression for **s** given by (2),

$$R = \frac{e^2}{16\,\pi^2 c^4}\left\{f^2 \int \frac{d\sigma}{r^2(1-\beta\cos\theta)^3} + 2\beta \int \frac{f_r f_x d\sigma}{r^2(1-\beta\cos\theta)^4} - (1-\beta^2)\int \frac{f_r^2 d\sigma}{r^2(1-\beta\cos\theta)^5}\right\}$$

$$= \frac{e^2}{6\,\pi c^3}\left\{\frac{f_x^2}{(1-\beta^2)^3} + \frac{f_y^2 + f_z^2}{(1-\beta^2)^2}\right\}. \quad (3)$$

If **f**$'$ is the acceleration of the electron at the time t relative to the system in which it is, at that instant, momentarily at rest, reference to equations (27), (28), (29), section 6, shows that

$$R = \frac{e^2 f'^2}{6\,\pi c^3}. \quad (4)$$

34. Radiation from a group of electrons. In the calculations of this section it will be assumed that the greatest distance between two electrons of the group is small compared with the wave length of the radiation emitted, and that the velocity of the fastest moving electron of the group is small compared with

that of light. If a sphere of radius r, large compared with the dimensions of the group, is described about some point in the group as center, the energy passing through the surface of this sphere in unit time is found as in the last article to be

$$R=\int \mathbf{s}\cdot d\boldsymbol{\sigma}, \tag{5}$$

where

$$\begin{aligned}\mathbf{s}&=E^2\mathbf{c}\\&=\mathbf{c}\sum_{ij}\mathbf{E}_i\cdot\mathbf{E}_j,\end{aligned}$$

the summation being taken over the n electrons in the group.

Now
$$E_i=\frac{e\,\{\mathbf{f}_i\cdot\mathbf{c}\mathbf{c}-c^2\mathbf{f}_i\}}{4\,\pi r c^4} \tag{6}$$

at a great distance from the ith electron.

Therefore
$$\mathbf{s}=\frac{e^2\mathbf{c}}{16\,\pi^2r^2c^6}\sum_{ij}\{c^2\mathbf{f}_i\cdot\mathbf{f}_j-\mathbf{f}_i\cdot\mathbf{c}\mathbf{f}_j\cdot\mathbf{c}\}. \tag{7}$$

Suppose that the sum of the components of the accelerations in any direction is equal to that in the opposite direction. Then $\mathbf{s}$ vanishes for all directions of $\mathbf{c}$. Hence a ring of any number of evenly spaced electrons which are rotating about a common axis with constant speed, will emit no appreciable radiation.

To get the total radiation from the group of electrons, substitute (7) in (5) and integrate. In this way it is found that

$$R=\frac{e^2}{6\,\pi c^3}\sum_{ij}\mathbf{f}_i\cdot\mathbf{f}_j. \tag{8}$$

35. Energy of a moving electron. The dynamical equation of an electron has been shown to be

$$\mathbf{K}=m\mathbf{f}-n\dot{\mathbf{f}},$$

through terms of the third order, where

$$m\equiv\frac{e^2}{6\,\pi a c^2},$$

$$n\equiv\frac{e^2}{6\,\pi c^3}.$$

The work done on the electron in a time t by the impressed field is

$$\begin{aligned} W &= \int \mathbf{K}\cdot\mathbf{v}dt \\ &= m\int \mathbf{f}\cdot\mathbf{v}dt - n\int \dot{\mathbf{f}}\cdot\mathbf{v}dt \\ &= \frac{1}{2}m\,(v_2^2 - v_1^2) - n(\mathbf{v}_2\cdot\mathbf{f}_2 - \mathbf{v}_1\cdot\mathbf{f}_1) + \frac{e^2}{6\,\pi c^3}\int f^2 dt, \end{aligned} \tag{9}$$

where the subscript 1 denotes the value of the quantity to which it is attached at the beginning of the time t and the subscript 2 the value of this quantity at the end of this time. The first term in this expression represents the kinetic energy of the electron, the second term its acceleration energy, and the third term the energy which has been radiated. The first two terms are reversible in the sense that the energy which they represent may be recovered when the electron returns to its original state of motion, but the third is irreversible.

Consider an electron which starts from rest and acquires a velocity $\mathbf{v}$ by virtue of a very small acceleration continued for a very long time. By making the derivatives of the acceleration small enough, the second and succeeding terms of the dynamical equation may be made as small as desired compared with the first term. Hence, if account is taken of the variation of mass with velocity, this equation is given to any desired degree of accuracy by

$$K = \frac{m}{(1-\beta^2)^{\frac{3}{2}}}\mathbf{f},$$

and the work done is expressed as closely as may be desired by

$$\begin{aligned} W &= \int \mathbf{K}\cdot\mathbf{v}dt \\ &= mc^2\left\{\frac{1}{\sqrt{1-\beta^2}} - 1\right\} \\ &= \frac{e^2}{6\,\pi a}\left\{\frac{1}{\sqrt{1-\beta^2}} - 1\right\}. \end{aligned} \tag{10}$$

In this case the energy radiated is inappreciable, so all the energy acquired by the electron may be recovered if it is brought back to its original state of rest by a similar process.

Now the energy of an electron moving with constant velocity **v** is given by

$$U = \frac{1}{2}\int (E^2 + H^2)\, d\tau.$$

Inside the electron's surface both **E** and **H** vanish, while outside the surface the values of these intensities are given by equations (3) and (4), section 19. Hence if the angle which the radius vector makes with the direction of the electron's velocity is denoted by θ,

$$U = \frac{e^2(1-\beta^2)^2}{16\,\pi}\iint \frac{1+\beta^2\sin^2\theta}{r^2(1-\beta^2\sin^2\theta)^3}\sin\theta d\theta dr,$$

where r goes from

$$\frac{a\sqrt{1-\beta^2}}{\sqrt{1-\beta^2\sin^2\theta}}$$

to infinity, and θ from 0 to π.

Integrating, $$U = \frac{e^2}{6\,\pi a}\left\{\frac{1}{\sqrt{1-\beta^2}} - \frac{1}{4}\sqrt{1-\beta^2}\right\}, \tag{11}$$

and the increase in energy is given by

$$U - U_0 = \frac{e^2}{6\,\pi a}\left\{\frac{1}{\sqrt{1-\beta^2}} - \frac{1}{4}\sqrt{1-\beta^2} - \frac{3}{4}\right\}. \tag{12}$$

The discrepancy between equations (10) and (12) arises from the fact that in the calculation from which the former was obtained no account was taken of the work done against the electron's field in connection with the progressive contraction which takes place as its velocity increases. In order to determine the work done in this process of contraction, it is necessary to evaluate the stress **K** on each unit area of the electron's surface. From the expressions for the electric and magnetic

intensities given in section 19 it follows that the electromagnetic force per unit volume just outside the surface is given by

$$F_x = \rho E_x$$
$$= \frac{\rho e}{4\pi r^2}\frac{(1-\beta^2)\cos\theta}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}},$$
$$F_y = \rho\{E_y - \beta H\}$$
$$\doteq \frac{\rho e}{4\pi r^2}\frac{(1-\beta^2)^2\sin\theta}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}},$$

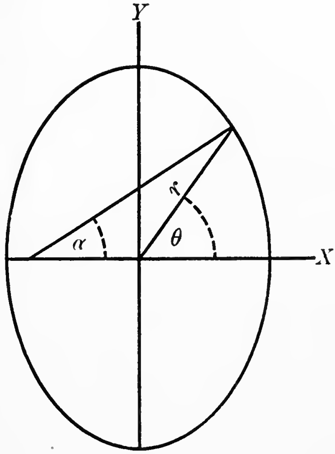

Fig. 13

where the X axis is taken in the direction of the electron's velocity. If α (Fig. 13) is the angle which the normal to the surface makes with the X axis,

$$\tan\alpha = (1-\beta^2)\tan\theta.$$

Therefore the tangential component F_t of $\mathbf{F}$ is given by

$$F_t = -F_x\sin\alpha + F_y\cos\alpha$$
$$= 0, \tag{13}$$

and the normal component F_n by

$$F_n = F_x\cos\alpha + F_y\sin\alpha$$
$$= \frac{\rho e}{4\pi r^2}\frac{(1-\beta^2)\sqrt{\cos^2\theta + (1-\beta^2)^2\sin^2\theta}}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}. \tag{14}$$

Now the charge per unit area is easily shown to be

$$\rho = \frac{e}{4\pi r^2}\frac{(1-\beta^2)}{\sqrt{1-\beta^2\sin^2\theta}\sqrt{\cos^2\theta+(1-\beta^2)^2\sin^2\theta}}. \tag{15}$$

Therefore, as the electromagnetic force just inside the electron's surface is zero,

$$K = \frac{1}{2}(F_n + 0)$$
$$= \frac{e^2}{32\pi^2 r^4}\frac{(1-\beta^2)^2}{(1-\beta^2\sin^2\theta)^2}$$
$$= \frac{e^2}{32\pi^2 a^4}. \tag{16}$$

As this stress has the nature of a hydrostatic tension independent of the velocity of the electron, Poincaré has been led to suggest that the electron may be held together by an equal and opposite hydrostatic pressure of an extra-electrodynamical character.

As the electron's velocity increases from 0 to v, its volume decreases from

$$\tfrac{4}{3}\pi a^3$$

to

$$\tfrac{4}{3}\pi a^3\sqrt{1-\beta^2}.$$

Therefore the work done against the stress of (16) is

$$W=\frac{e^2}{6\pi a}\left\{\frac{1}{4}-\frac{1}{4}\sqrt{1-\beta^2}\right\}. \tag{17}$$

Adding this to (10), (12) is obtained.

36. Diffraction of X Rays. Equation (3) of section 33 shows that in general an electron will radiate energy whenever it is accelerated. However irregular its motion may be, the radiation emitted may be analyzed by Fourier's method into a series of superimposed simple harmonic waves. Waves of a length from 4000 Å to 8000 Å constitute light of the visible spectrum, whereas waves of a length of the order of 1 Å are called X rays. These rays have great penetrating power, and all attempts to diffract them were unsuccessful until Laue suggested in 1913 that the distances between adjacent atoms in crystals were of such a magnitude as to make these substances suitable natural gratings for the diffraction of X rays. The following theory is presented very nearly in the form given originally by Laue.

Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ be vectors having the lengths and directions of the edges of an elementary parallelepiped of the crystal. Then if x, y, z are the coördinates of an atom relative to an origin O at the center of the crystal,

$$\begin{aligned} x &= ma_{1x}+na_{2x}+pa_{3x}, \\ y &= ma_{1y}+na_{2y}+pa_{3y}, \\ z &= ma_{1z}+na_{2z}+pa_{3z}, \end{aligned} \tag{18}$$

where m, n, p are positive or negative integers. Let r be the distance of this atom from the observer at P, and R the distance

of the center of the crystal O from P. These two distances will, in general, be so nearly equal that they may be considered the same, except in so far as the phase of the radiation is concerned.

Let the incident radiation be plane, the direction cosines of the wave normal being denoted by α_0, β_0, γ_0. Then if the intra-atomic vibrators are all alike, the displacement of any vibrator at a time t will be given by the real part of

$$A_0 e^{i\frac{2\pi}{\lambda}(x\alpha_0+y\beta_0+z\gamma_0-ct)},$$

and the field strength at P due to this displacement by

$$\frac{A}{R} e^{i\frac{2\pi}{\lambda}(r+x\alpha_0+y\beta_0+z\gamma_0-ct)},$$

since the electric intensity at a distance from the atom great compared with the wave length varies directly as the acceleration of the vibrator, which is proportional to its displacement, and inversely as the distance. The coefficient A is a function of the direction cosines α, β, γ, of the line OP, as well as of α_0, β_0, γ_0.

The total electric intensity at P, then, is equal to

$$E = \frac{\sum A e^{i\frac{2\pi}{\lambda}(r+x\alpha_0+y\beta_0+z\gamma_0-ct)}}{R}$$

$$\doteq \frac{A}{R} e^{i\frac{2\pi}{\lambda}(R-ct)} \sum e^{i\frac{2\pi}{\lambda}\{x(\alpha_0-\alpha)+y(\beta_0-\beta)+z(\gamma_0-\gamma)\}}, \qquad (19)$$

since $$r \doteq R-(x\alpha+y\beta+z\gamma)$$

is a sufficiently close approximation for the exponent.

Put $$F \equiv \frac{2\pi}{\lambda}\{a_{1x}(\alpha_0-\alpha)+a_{1y}(\beta_0-\beta)+a_{1z}(\gamma_0-\gamma)\},$$

$$G \equiv \frac{2\pi}{\lambda}\{a_{2x}(\alpha_0-\alpha)+a_{2y}(\beta_0-\beta)+a_{2z}(\gamma_0-\gamma)\},$$

$$H \equiv \frac{2\pi}{\lambda}\{a_{3x}(\alpha_0-\alpha)+a_{3y}(\beta_0-\beta)+a_{3z}(\gamma_0-\gamma)\}.$$

Then, by virtue of (18),

$$E=\frac{A}{R}e^{i\frac{2\pi}{\lambda}(R-ct)}\sum_m\sum_n\sum_p e^{i(mF+nG+pH)}$$
$$=\frac{A}{R}e^{i\frac{2\pi}{\lambda}(R-ct)}\sum_m e^{imF}\sum_n e^{inG}\sum_p e^{ipH}. \tag{20}$$

Hence the intensity of the diffracted radiation at P is maximum for those directions for which

$$\sum_m e^{imF},\ \sum_n e^{inG},\ \sum_p e^{ipH}$$

have their greatest absolute values.

Suppose the illuminated portion of the crystal to be bounded by planes parallel to the sides of the elementary parallelepiped. Then m varies from $-M$ to $+M$, n from $-N$ to $+N$, and p from $-P$ to $+P$, where M, N, P are positive integers. Hence

$$\sum_m e^{imF}=1+e^{iF}+e^{2iF}+\cdots+e^{MiF}$$
$$+e^{-iF}+e^{-2iF}+\cdots+e^{-MiF}$$
$$=1+2\{\cos F+\cos 2F+\cdots+\cos MF\}.$$

The absolute value of this expression is obviously a maximum when

$$F=\pm 2q_1\pi,$$

where q_1 is a positive integer.

Therefore the conditions for maximum intensity are

$$\begin{aligned} a_{1x}\alpha+a_{1y}\beta+a_{1z}\gamma &= a_{1x}\alpha_0+a_{1y}\beta_0+a_{1z}\gamma_0\pm q_1\lambda,\\ a_{2x}\alpha+a_{2y}\beta+a_{2z}\gamma &= a_{2x}\alpha_0+a_{2y}\beta_0+a_{2z}\gamma_0\pm q_2\lambda,\\ a_{3x}\alpha+a_{3y}\beta+a_{3z}\gamma &= a_{3x}\alpha_0+a_{3y}\beta_0+a_{3z}\gamma_0\pm q_3\lambda, \end{aligned} \tag{21}$$

where q_2 and q_3 are also positive integers.

The left-hand member of each of these equations is proportional to the cosine of the angle between one of the sides of the elementary parallelepiped and the diffracted ray, while the right-hand side is equal to an integral number of wave lengths plus a quantity proportional to the cosine of the angle between the same side of this parallelepiped and the incident ray. So for each value of q_1, q_2, and q_3 these equations define three cones in α, β, and γ. The loci of maximum intensity will be the lines of intersection of these cones.

Consider the simple case where $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are mutually perpendicular and equal. Choose axes parallel to the sides of the elementary parallelepiped, and consider an incident wave advancing along the X axis. Then the conditions contained in (21) reduce to

$$\begin{aligned} a\alpha &= a \pm q_1\lambda, \\ a\beta &= \pm q_2\lambda, \\ a\gamma &= \pm q_3\lambda. \end{aligned} \tag{22}$$

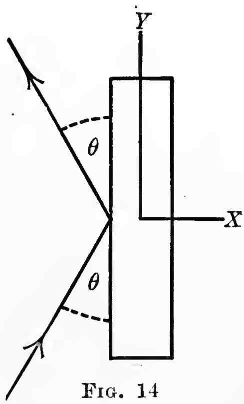

FIG. 14

If a screen is placed at right angles to the X axis on the side of the crystal opposite to the source, the intensity of the transmitted radiation will be greatest along the traces on the screen of the cones defined by (22). The trace on the screen of the cone defined by the first of these equations is a circle, while those defined by the second and third are hyperbolæ. The greatest intensity of all will be produced at those points on the screen where two, or better three, of these plane curves intersect.

Consider radiation incident on the crystal under consideration at a small glancing angle θ (Fig. 14). If the wave normal is in the XY plane,

$$\alpha_0 = \sin\theta,$$
$$\beta_0 = \cos\theta,$$
$$\gamma_0 = 0,$$

and

$$a\alpha = a\sin\theta \pm q_1\lambda,$$
$$a\beta = a\cos\theta \pm q_2\lambda,$$
$$a\gamma = \pm q_3\lambda.$$

Consider the reflected radiation. For this

$$\alpha = -\sin\theta,$$
$$\beta = \cos\theta,$$
$$\gamma = 0,$$

and

$$2a\sin\theta = \pm q_1\lambda, \tag{23}$$
$$q_2 = 0,$$

for reinforcement.

CHAPTER VII

ELECTROMAGNETIC FIELDS IN MATERIAL MEDIA

37. Fundamental equations. Consider a medium composed of positive and negative electrons. Experimental observation of the field in such a medium is limited, by the coarseness of available instruments, to the investigation of the average values of the electric and magnetic intensities. These average values are defined in the following way. Divide the medium up into fixed elements of volume τ so large that each contains very many electrons, but yet so small that no measuring instrument which may be used to investigate the field can detect a variation of electrical properties from one point to another in any one of these elements. Then the average value at any point of a scalar ϕ which depends upon the state of the electrons in the medium is defined by

$$\bar{\phi} \equiv \frac{1}{\tau}\int \phi d\tau,$$

where the integral is taken through the volume element τ. Similarly, the average value of a vector $\mathbf{V}$ is defined by

$$\overline{\mathbf{V}} \equiv \frac{1}{\tau}\int \mathbf{V} d\tau.$$

In the absence of an impressed field, each volume element of the medium will be supposed to contain equal numbers of positive and negative electrons moving about in a fortuitous manner. Consequently the average charge and average current will vanish. In the presence of a field, however, the electric intensity may cause electrons of opposite sign to be displaced in opposite directions, with the result that the average density of charge may no longer be everywhere zero. In the same way, the magnetic intensity may orient intra-atomic rings of electrons in such a way as to produce an average current different from

zero. Analytical difficulties involved in discontinuities in the medium may be avoided by imagining every charged surface to be replaced by a region of finite but very small thickness, in which ρ varies rapidly but continuously from one value to another.

As the elements of volume τ are fixed,

$$\overline{\frac{\partial\phi}{\partial x}} \equiv \frac{1}{\tau}\int_\tau \frac{\partial\phi}{\partial x}d\tau$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{\tau}\int_\tau \phi d\tau\right\}$$

$$= \frac{\partial}{\partial x}\overline{\phi},$$

$$\overline{\frac{\partial\phi}{\partial t}} = \frac{\partial}{\partial t}\overline{\phi},$$

etc., and similarly for the vector function $\mathbf{V}$.

To find the average value of ρ, consider a volume element τ of dimensions Δx, Δy, Δz. In the time dt the charge entering τ through those sides of this volume element which are perpendicular to the X axis is

$$\overline{\rho v_x}\Delta y\Delta z dt - \left\{\overline{\rho v_x} + \frac{\partial}{\partial x}(\overline{\rho v_x})\,\Delta x\right\}\Delta y\Delta z dt$$

$$= -\frac{\partial}{\partial x}(\overline{\rho v_x})\,\Delta x\Delta y\Delta z dt.$$

Therefore, taking the six sides of τ into account,

$$\frac{\partial}{\partial t}\overline{\rho} = -\nabla\cdot(\overline{\rho\mathbf{v}}).$$

Put
$$\mathbf{Q} \equiv \int \overline{\rho\mathbf{v}}dt.$$

Then
$$\overline{\rho} = -\nabla\cdot\mathbf{Q}, \tag{1}$$

and
$$\overline{\rho\mathbf{v}} = \dot{\mathbf{Q}}. \tag{2}$$

Hence, for a material medium, the equations of the electromagnetic field given in section 26 take the following form:

$$\nabla\cdot(\overline{\mathbf{E}} + \mathbf{Q}) = 0, \quad (3) \qquad \nabla\cdot\overline{\mathbf{H}} = 0, \quad (5)$$

$$\nabla\times\overline{\mathbf{E}} = -\frac{1}{c}\dot{\overline{\mathbf{H}}}, \quad (4) \qquad \nabla\times\overline{\mathbf{H}} = \frac{1}{c}(\dot{\overline{\mathbf{E}}} + \dot{\mathbf{Q}}). \quad (6)$$

The method used in deriving these equations has not been such as to limit their applicability to media which are homogeneous or isotropic. They apply to all media made up of electrons whatever their nature. Moreover, as two media in contact may be considered equivalent to a single non-homogeneous medium containing a thin transition layer in which the properties of the medium change rapidly but continuously, these equations may be applied in the region of contact.

In order to determine the relation between $\mathbf{Q}$ on the one hand and $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ on the other, it is necessary to consider the motion of individual electrons in the medium. These electrons consist of two classes: (a) the free electrons, which move among the atoms, and (b) the bound electrons, whose displacements are limited by the boundaries of the atoms to which they belong.

As an electron is of very small dimensions compared with an atom, the number of times a free electron collides with another electron is negligible compared with the number of times it collides with an atom. Hence if an electron strikes against an atom ν times a second, its average drift velocity is given by

$$\overline{\mathbf{v}} = \frac{\overline{\mathbf{f}}}{2\nu},$$

where $\overline{\mathbf{f}}$ is the average acceleration produced by the intensities $\mathbf{E}_1$ and $\mathbf{H}_1$ due to all charges other than that on the electron under consideration. Now

$$m\mathbf{f} = e\left\{\mathbf{E}_1 + \frac{1}{c}\mathbf{v} \times \mathbf{H}_1\right\}$$

approximately, and therefore if v is small compared to c the current due to the free electrons is given by

$$\overline{\rho\mathbf{v}} = \frac{Ne^2}{2m\nu}\overline{\mathbf{E}}_1, \tag{7}$$

where N is the number of free electrons per unit volume.

Describe a sphere of volume $1/N$ around the electron under consideration. Then if $\mathbf{E}_2$ is the average electric intensity within this sphere due to this electron,

$$\overline{\mathbf{E}}_1 = \overline{\mathbf{E}} - \overline{\mathbf{E}}_2.$$

In general $\overline{\mathbf{E}}_2$ is small compared to the other quantities involved in this equation. Hence (7) may be written

$$\overline{\rho\mathbf{v}} = \frac{Ne^2}{2\,m\nu}\overline{\mathbf{E}}$$

$$= C\overline{\mathbf{E}}, \tag{8}$$

where C is the conductivity.

A bound electron is supposed to be held in the atom to which it belongs by a force of restitution proportional to its displacement $\mathbf{R}$ from the center of the atom. In addition, it is supposed to be subject to a frictional resistance proportional to its velocity. Hence, if $\mathbf{E}_1$ is the electric intensity and $\mathbf{H}_1$ the magnetic intensity due to all causes external to the atom under consideration, the equation of motion of a bound electron inside this atom is

$$m\ddot{\mathbf{R}} + \alpha\dot{\mathbf{R}} + \beta\mathbf{R} = e\left\{\mathbf{E}_1 + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{H}_1\right\},$$

where those terms in the dynamical reaction other than the one due to the mass have been omitted as negligible.

Put
$$2\,l \equiv \frac{\alpha}{m},$$

$$k_0^2 \equiv \frac{\beta}{m}.$$

Then the equation of motion becomes

$$\ddot{\mathbf{R}} + 2\,l\dot{\mathbf{R}} + k_0^2\mathbf{R} = \frac{e}{m}\left\{\mathbf{E}_1 + \frac{1}{c}\dot{\mathbf{R}} \times \mathbf{H}_1\right\}. \tag{9}$$

The displacement $\mathbf{R}$ satisfying this equation consists of the solution $\mathbf{R}_0$ of the complementary equation

$$\ddot{\mathbf{R}}_0 + 2\,l\dot{\mathbf{R}}_0 + k_0^2\mathbf{R}_0 = \frac{e}{mc}\dot{\mathbf{R}}_0 \times \mathbf{H}_1 \tag{10}$$

added to a particular solution $\mathbf{R}_p$ of the original equation.

Obviously $\mathbf{R}_p$ represents the part of the displacement produced by the electric field $\mathbf{E}_1$, while $\mathbf{R}_0$ is the part due to the

natural motions of the electrons inside the atom as modified by the presence of the magnetic field $\mathbf{H}_1$. Since $\dot{\mathbf{R}}_p$ is in general negligible compared to c, the particular solution may be obtained to a sufficient degree of accuracy from the equation

$$\ddot{\mathbf{R}}_p + 2\,l\dot{\mathbf{R}}_p + k_0^2\mathbf{R}_p = \frac{e}{m}\mathbf{E}_1. \tag{11}$$

Now if there are N atoms per unit volume, and each of these contains n electrons, the current produced by the bound electrons is given by

$$\overline{\rho\mathbf{v}} = Nn\overline{e\dot{\mathbf{R}}_p} + Nn\overline{e\dot{\mathbf{R}}_0}.$$

While $e\mathbf{R}_0$ may vary greatly from one electron to the next, it is evident that $e\mathbf{R}_p$ will have more or less the direction of the average electric intensity. Consequently the contribution to

$$\overline{e\dot{\mathbf{R}}_p}$$

due to the entrance of new electrons into the volume element τ through which the average is being taken will be negligible compared to the part of this quantity dependent upon the electrons already inside this region. Therefore

$$\overline{e\dot{\mathbf{R}}_p} \doteq \dot{\overline{e\mathbf{R}_p}},$$

and

$$\overline{\rho\mathbf{v}} = Nn\dot{\overline{e\mathbf{R}_p}} + Nn\overline{e\dot{\mathbf{R}}_0}. \tag{12}$$

If the electrons inside the atom are subject to different restoring and resisting forces, this equation must be replaced by

$$\overline{\rho\mathbf{v}} = N\sum_i n_i\dot{\overline{e\mathbf{R}_p}} + N\sum_i n_i\overline{e\dot{\mathbf{R}}_0}, \tag{13}$$

where n_i denotes the number of electrons of type i inside each atom.

The *electric polarization*, or *electric moment per unit volume*, is defined as

$$\mathbf{P} \equiv Nn\overline{e\mathbf{R}_p},$$

or, if each atom contains more than one type of electron,

$$\mathbf{P} \equiv N\sum_i n_i\overline{e\mathbf{R}_p}.$$

Hence (13) may be written

$$\overline{\rho \mathbf{v}} = \dot{\mathbf{P}} + N\sum_i n_i \overline{e\dot{\mathbf{R}}_0}. \tag{14}$$

At first sight it would seem as though $\overline{e\dot{\mathbf{R}}_0}$ must always vanish. In a magnetic substance, however, this is not necessarily the case, as will now be shown.

Suppose that for certain types of electrons the damping coefficient l in equation (10) is zero, and that the motion is constrained to a plane. Then motion in a circle with constant frequency is a solution of this equation. A substance each of whose atoms contains one or more rings of electrons revolving about their respective axes is said to be *magnetic.* The electrons in each ring will be assumed to be evenly spaced, and to have a constant angular velocity $\mathbf{\Omega}$ about the axis of the ring so long as the external magnetic field remains unchanged.

In the presence of a magnetic field each electron will be subject to a force which will tend, in general, to change the radius R_0 of the ring and the frequency of revolution. Moreover, the ring as a whole will be acted on by a torque which will tend to orient it. The force on an electron resulting from the magnetic intensity $\mathbf{H}_1$ due to all causes external to the atom in which it lies is given by

$$\begin{aligned}\mathbf{K} &= \frac{e}{c}(\mathbf{v} \times \mathbf{H}_1)\\ &= \frac{e}{c}(\mathbf{\Omega} \times \mathbf{R}_0) \times \mathbf{H}_1,\end{aligned}$$

as $\mathbf{R}_0$ is measured from the center of the ring. The torque about this point is

$$\begin{aligned}\mathbf{G} &= \sum \mathbf{R}_0 \times \mathbf{K}\\ &= \frac{e}{c}\sum \mathbf{R}_0\cdot\mathbf{H}_1 \mathbf{\Omega} \times \mathbf{R}_0\end{aligned}$$

for the entire ring. The form of this expression shows that $\mathbf{G}$ is perpendicular to $\mathbf{\Omega}$. Denote by ϕ the angle which $\mathbf{R}_0$ makes

with the plane of $\mathbf{\Omega}$ and $\mathbf{H}_1$. Then if θ is the angle between $\mathbf{\Omega}$ and $\mathbf{H}_1$, the component $G_{\parallel}$ of the torque parallel to the plane of these two vectors is

$$G_{\parallel} = \frac{e}{c}\sum R_0^2 \Omega H_1 \sin\theta \sin\phi \cos\phi$$
$$= 0,$$

and the component $G_{\perp}$ at right angles to this plane is

$$G_{\perp} = \frac{e}{c}\sum R_0^2 \Omega H_1 \sin\theta \cos^2\phi$$
$$= \frac{neR_0^2}{2\,c}\Omega H_1 \sin\theta,$$

where n is the number of electrons in the ring. The quantity

$$\mathbf{M} \equiv \frac{neR_0^2}{2\,c}\mathbf{\Omega}$$

is called the *magnetic moment* of the ring. Therefore the torque is given by

$$\mathbf{G} = \mathbf{M} \times \mathbf{H}_1. \tag{15}$$

Denote by dN the number of atoms per unit volume in which the axes of the rings of electrons have a given direction. Then the *intensity of magnetization*, or *magnetic moment per unit volume*, is defined as

$$\mathbf{I} \equiv N\overline{\mathbf{M}}$$
$$= \int \mathbf{M}\,dN,$$

or, if each atom contains more than one ring,

$$\mathbf{I} \equiv N\sum_i \overline{\mathbf{M}}_i.$$

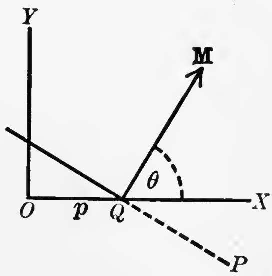

FIG. 15

Consider a ring of electrons whose center is located on the X axis (Fig. 15) at a distance p to the right of the origin. Denote by θ the angle which $\mathbf{M}$ makes with the X axis, and by ϕ the angle which the MX plane makes with the XY plane. It is desired to find the value of

$$\sum e\dot{\mathbf{R}}_0$$

due to electrons in this ring which lie to the right of the YZ coördinate plane. Evidently this sum will differ from zero only if the ring is cut by this plane; that is, if

$$-R_0 \sin\theta < p < R_0 \sin\theta.$$

Moreover, it is obvious that this sum will have no component in the X direction. If ψ is the angle between $\mathbf{R}_0$ and the intersection $\overline{QP}$ of the plane of the ring with the MX plane,

$$\left(\sum e\dot{R}_0\right)_y = \sin\phi \sum e\dot{R}_0 \cos\psi$$

$$= \frac{ne\dot{R}_0}{2\pi} \sin\phi \int \cos\psi d\psi$$

integrated from $$-\cos^{-1}\left(\frac{p}{R_0 \sin\theta}\right)$$

to $$\cos^{-1}\left(\frac{p}{R_0 \sin\theta}\right).$$

Therefore $$\left(\sum e\dot{R}_0\right)_y = \frac{ne\Omega}{\pi} \frac{\sqrt{R_0^2 \sin^2\theta - p^2}}{\sin\theta} \sin\phi.$$

Summing up over a prism of unit cross section extending from

$$p = -R_0 \sin\theta$$

to $$p = R_0 \sin\theta,$$

the y component of the current to the right of the YZ plane due to all rings which are cut by a unit area of this plane is found to be

$$\sum\left(\sum e\dot{R}_0\right)_y = \frac{ne\Omega}{\pi} \int \frac{\sin\phi}{\sin\theta} dN \int \sqrt{R_0^2 \sin^2\theta - p^2}\, dp$$

$$= c\int M \sin\theta \sin\phi dN$$

$$= c\int M_z dN$$

$$= cI_z.$$

Similarly, $$\sum\left(\sum e\dot{R}_0\right)_z = -cI_y,$$

and, in general, $$\sum\left(\sum e\dot{\mathbf{R}}_0\right) = c\mathbf{I} \times \mathbf{n}, \tag{16}$$

where $\mathbf{n}$ is a unit vector normal to the surface on the positive side of which the current is to be computed.

Consider a volume element τ of dimensions Δx, Δy, Δz. The current contained in it on account of those rings of electrons which are cut by the sides perpendicular to the X axis is

$$c\mathbf{I} \times \mathbf{i}\Delta y \Delta z - c\left(\mathbf{I} + \frac{\partial \mathbf{I}}{\partial x}\Delta x\right) \times \mathbf{i}\Delta y \Delta z$$

$$= c\left\{-\mathbf{j}\frac{\partial I_z}{\partial x} + \mathbf{k}\frac{\partial I_y}{\partial x}\right\}\Delta x \Delta y \Delta z.$$

Therefore, taking the six sides of τ into account,

$$Nne\overline{\dot{\mathbf{R}}_0} = c\nabla \times \mathbf{I}, \tag{17}$$

or, if each atom contains more than one ring of electrons,

$$N\sum_i n_i e\overline{\dot{\mathbf{R}}_0} = c\nabla \times \mathbf{I}. \tag{18}$$

Hence the entire current due to bound electrons is given by

$$\overline{\rho \mathbf{v}} = \dot{\mathbf{P}} + c\nabla \times \mathbf{I}. \tag{19}$$

Consider a medium in which there may exist currents due both to conduction by free electrons and to the displacement of bound electrons. Then it follows from (8) and (19) that

$$\dot{\mathbf{Q}} = C\overline{\mathbf{E}} + \dot{\mathbf{P}} + c\nabla \times \mathbf{I}, \tag{20}$$

and equation (6) becomes

$$\nabla \times (\overline{\mathbf{H}} - \mathbf{I}) = \frac{1}{c}(\dot{\overline{\mathbf{E}}} + \dot{\mathbf{P}} + C\overline{\mathbf{E}}). \tag{21}$$

In accord with the usual practice put

$$\mathbf{D} \equiv \overline{\mathbf{E}} + \mathbf{P},$$

and

$$\mathbf{B} \equiv \overline{\mathbf{H}}.$$

The vector $\mathbf{D}$ is called the *electric displacement*, and $\mathbf{B}$ the *magnetic induction.* The average magnetic intensity due to all causes other than the intensity of magnetization $\mathbf{I}$ of the medium is usually denoted by $\mathbf{H}$. As this letter has been used with another meaning in the preceding pages, this quantity will be designated by $\mathbf{L}$. Then, omitting strokes, equation (6) gives

$$\nabla \times \mathbf{L} = \frac{1}{c}(\dot{\mathbf{E}} + \dot{\mathbf{P}} + C\mathbf{E}),$$

and comparison with (21) shows that

$$\mathbf{L} = \mathbf{B} - \mathbf{I},$$

since $\mathbf{L}$ vanishes with $\mathbf{B}$ and $\mathbf{I}$. As $\mathbf{B}$ is the total average magnetic intensity, $\mathbf{I}$ represents the average magnetic intensity produced by the magnetization of the medium. Equation (21) may now be written

$$\nabla \times \mathbf{L} = \frac{1}{c}(\dot{\mathbf{D}} + C\mathbf{E}). \tag{22}$$

Returning to (20), it follows that

$$\nabla \cdot \dot{\mathbf{Q}} = C\nabla \cdot \overline{\mathbf{E}} + \nabla \cdot \dot{\mathbf{P}},$$

and

$$\nabla \cdot \mathbf{Q} = C\int \nabla \cdot \overline{\mathbf{E}}dt + \nabla \cdot \mathbf{P}, \tag{23}$$

as $\mathbf{Q}$ and $\mathbf{P}$ vanish everywhere when no external field is present. Therefore, omitting strokes, equation (3) becomes

$$\nabla \cdot \mathbf{D} = -C\int \nabla \cdot \mathbf{E}dt. \tag{24}$$

Hence the equations of the electromagnetic field in a medium containing both free and bound electrons take the form

$$\nabla \cdot \mathbf{D} = -C\int \nabla \cdot \mathbf{E}dt, \quad (25) \qquad \nabla \cdot \mathbf{B} = 0, \quad (27)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (26) \qquad \nabla \times \mathbf{L} = \frac{1}{c}(\dot{\mathbf{D}} + C\mathbf{E}), \quad (28)$$

when the field is investigated from the macroscopic point of view. As already noted, these equations apply whether or not the medium is homogeneous or isotropic, and they are valid in the region of contact of two media.

If, in addition to the processes of conduction and displacement, charges ρ and currents $\rho\mathbf{v}$ are produced by convection through the medium, the electromagnetic equations for this most general case assume the form

$$\nabla \cdot \mathbf{D} = -C\int \nabla \cdot \mathbf{E}dt + \rho, \quad (29) \qquad \nabla \cdot \mathbf{B} = 0, \quad (31)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (30) \qquad \nabla \times \mathbf{L} = \frac{1}{c}(\dot{\mathbf{D}} + C\mathbf{E} + \rho\mathbf{v}). \quad (32)$$

38. Specific inductive capacity. In order to obtain the relation between **P** and $\overline{\mathbf{E}}$, and hence between **D** and $\overline{\mathbf{E}}$, it is necessary to solve equation (11) of the preceding section. The following discussion will be confined to the case of simple harmonic fields, a steady field being considered as a simple harmonic field of zero frequency. The electric and magnetic intensities are given by the real parts of the following complex quantities,

$$\mathbf{E} = \mathbf{E}_0 e^{-i\omega t},$$

$$\mathbf{H} = \mathbf{H}_0 e^{-i\omega t},$$

and

$$\frac{d}{dt} = -i\omega.$$

Hence the solution of (11) is

$$\mathbf{R}_p = \frac{\dfrac{e}{m}}{k_0^2 - \omega^2 - 2\,i\omega l}\mathbf{E}_1,$$

and **P** is given by

$$\mathbf{P} = \frac{Nn\dfrac{e^2}{m}}{k_0^2 - \omega^2 - 2\,i\omega l}\overline{\mathbf{E}}_1. \tag{33}$$

To find the relation between $\overline{\mathbf{E}}_1$ and $\overline{\mathbf{E}}$, describe about the center O (Fig. 16) of each atom a sphere of volume $1/N$. Then if $\mathbf{E}_2$ is the average electric intensity within one of these spheres due to the electrons which it contains,

$$\overline{\mathbf{E}}_1 = \overline{\mathbf{E}} - \overline{\mathbf{E}}_2.$$

Evidently $\mathbf{E}_2$ has the direction of the displacement $\mathbf{R}_p$ caused by the external field $\mathbf{E}_1$, and its magnitude is given approximately by

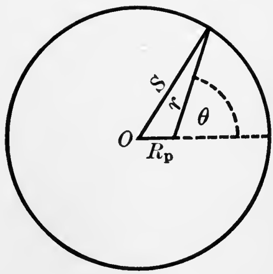

FIG. 16

$$E_2 = Nn\iint \frac{e}{4\,\pi r^2}\cos\theta\,(2\,\pi r^2 \sin\theta d\theta dr),$$

where θ is the angle which the radius vector makes with $\mathbf{R}_p$. As r goes from zero to

$$R_p\left\{-\cos\theta+\sqrt{\cos^2\theta+\frac{S^2}{R_p^2}-1}\right\},$$

it follows that
$$\mathbf{E}_2=-\tfrac{1}{3}Nne\mathbf{R}_p. \tag{34}$$

Therefore
$$\overline{\mathbf{E}}_1=\overline{\mathbf{E}}+\tfrac{1}{3}\mathbf{P},$$

and (33) becomes
$$\mathbf{P}=\frac{Nn\dfrac{e^2}{m}}{k^2-\omega^2-2\,i\omega l}\overline{\mathbf{E}}, \tag{35}$$

where
$$k^2\equiv k_0^2-\frac{1}{3}Nn\frac{e^2}{m}.$$

Put
$$\epsilon\equiv\frac{Nn\dfrac{e^2}{m}}{k^2-\omega^2-2\,i\omega l}.$$

Then, omitting strokes,
$$\mathbf{P}=\epsilon\mathbf{E}, \tag{36}$$

and
$$\mathbf{D}=(1+\epsilon)\mathbf{E}. \tag{37}$$

The *specific inductive capacity* κ is defined as the ratio of the displacement $\mathbf{D}$ to the electric intensity $\mathbf{E}$. Hence

$$\begin{aligned}\kappa&\equiv1+\epsilon\\&=1+\frac{Nn\dfrac{e^2}{m}}{k^2-\omega^2-2\,i\omega l},\end{aligned} \tag{38}$$

or, if each atom contains more than one type of electron,

$$\begin{aligned}\kappa&\equiv1+\sum_i\epsilon_i\\&=1+N\frac{e^2}{m}\sum_i\frac{n_i}{k_i^2-\omega^2-2\,i\omega l_i}.\end{aligned} \tag{39}$$

In the case of a steady field,
$$\kappa=1+\frac{Nne^2}{k^2m} \tag{40}$$

if there is only one kind of electron in each atom, and

$$\frac{Nne^2}{3\,k_0^2m}=\frac{\kappa-1}{\kappa+2}. \tag{41}$$

The number of atoms per unit volume varies directly with the density d. Therefore

$$\frac{\kappa - 1}{d(\kappa + 2)} \tag{42}$$

should be a constant. This relation has been verified for many gases.

The force of restitution in a polarized atom may be easily evaluated if n positive and n negative electrons are assumed to be distributed uniformly over equal and coincident spheres. If the center of one of these spheres remains stationary, while that of the other is displaced a distance $\mathbf{R}_p$, the force drawing them together is approximately

$$\frac{ne}{4\pi R_p^2}\left(\frac{\frac{4}{3}\pi R_p^3}{v} ne\right)$$
$$= \frac{1}{3}\frac{n^2e^2}{v} R_p,$$

where v is the volume of a sphere. Hence the force on each electron is

$$\frac{1}{3}\frac{ne^2}{v} R_p,$$

and

$$k_0^2 = \frac{1}{3}\frac{ne^2}{vm}.$$

Substituting in (41), it is found that

$$Nv = \frac{\kappa - 1}{\kappa + 2} \tag{43}$$

is the portion of the volume under consideration which is actually occupied by the atoms. If an estimate of N has been obtained from other sources, the atomic volume v may be computed. In this way the radius of an atom is found to be of the order of

$$(10)^{-8} \text{ cm}.$$

39. Magnetic permeability. It has been pointed out in section 37 that a magnetic field tends both to change the radius and the frequency of revolution of the ring of electrons inside a magnetic

atom, and to orient this ring. To determine the magnitude of the first effect, integrate both sides of the equation

$$\nabla \times \mathbf{E}_1 = -\frac{1}{c}\dot{\mathbf{H}}_1$$

over the surface bounded by a ring of electrons. Then

$$\int \nabla \times \mathbf{E}_1 \cdot d\sigma = -\frac{1}{c}\int \dot{\mathbf{H}}_1 \cdot d\sigma.$$

But by Stokes' theorem

$$\int \nabla \times \mathbf{E}_1 \cdot d\sigma = \int \mathbf{E}_1 \cdot d\lambda,$$

where the right-hand member is integrated around the ring. Hence the work done on the ring when the magnetic field is increased by $d\mathbf{H}_1$ is

$$\begin{aligned} dU &= \frac{ne\Omega}{2\pi} dt \int \mathbf{E}_1 \cdot d\lambda \\ &= -\frac{neR_0^2}{2c} \mathbf{\Omega} \cdot \dot{\mathbf{H}}_1 dt \\ &= -\mathbf{M} \cdot d\mathbf{H}_1. \end{aligned} \tag{44}$$

Now the increase in the kinetic and potential energies of the electrons in the ring is

$$\begin{aligned} dU &= nm(vdv + k_0^2 R_0 dR_0) \\ &= nmR_0\Omega(R_0 d\Omega + 2\,\Omega dR_0), \end{aligned}$$

as

$$k_0 = \Omega.$$

But

$$dM = \frac{neR_0}{2c}(R_0 d\Omega + 2\,\Omega dR_0).$$

Therefore

$$dU = \frac{2\,m\Omega c}{e} dM. \tag{45}$$

Equating this to the expression for the rate at which work is done on the ring,

$$dM = -\frac{e}{2\,m\Omega c} \mathbf{M} \cdot d\mathbf{H}_1. \tag{46}$$

Let α be the angle which $\mathbf{M}$ makes with $d\overline{\mathbf{H}}_1$. If initially the axes of as many atomic rings pointed in one direction as in any other, $d\mathbf{I}$ has the direction of $d\overline{\mathbf{H}}_1$ and is given in magnitude by

$$\begin{aligned} dI &= \int dM \cos\alpha \, dN \\ &= -\frac{eMd\overline{H}_1}{2\,m\Omega c}\int \cos^2\alpha \, dN \\ &= -\frac{NeM}{6\,m\Omega c}\, d\overline{H}_1. \end{aligned} \tag{47}$$

To find the relation between $d\overline{\mathbf{H}}_1$ and $d\overline{\mathbf{H}}$, describe about the center of each atom a sphere of volume $1/N$. Then if $\mathbf{H}_2$ is the average magnetic intensity within one of these spheres due to the electrons which it contains,

$$\overline{\mathbf{H}}_1 = \overline{\mathbf{H}} - \overline{\mathbf{H}}_2.$$

But it has been shown that the average magnetic intensity $\overline{\mathbf{H}}_2$ due to rings of electrons is equal to the intensity of magnetization $\mathbf{I}$. Therefore as

$$\mathbf{B} \equiv \overline{\mathbf{H}},$$

it follows that

$$\begin{aligned} \overline{\mathbf{H}}_1 &= \mathbf{B} - \mathbf{I} \\ &= \mathbf{L}, \end{aligned}$$

and

$$d\mathbf{I} = -\frac{NeM}{6\,m\Omega c}\, d\mathbf{L}. \tag{48}$$

The *permeability* μ of the medium is defined as the ratio of the magnetic induction B to the external field strength L. Therefore in the case under consideration

$$\mu = 1 - \frac{NeM}{6\,m\Omega c}. \tag{49}$$

The effect under discussion is known as *diamagnetism*, and is characterized by a value of the permeability less than unity. It is shown in the greatest degree by bismuth.

The tendency of a magnetic field to orient a ring of electrons will be considered next. The torque on such a ring has already been shown to be

$$\mathbf{G} = \mathbf{M} \times \mathbf{H}_1.$$

Let $\mathbf{H}$ be the average total magnetic intensity inside a sphere of volume $1/N$ described about the center of the atom under consideration, and let $\mathbf{H}_2$ be the average magnetic intensity within this sphere due to the electrons which it contains. Then

$$\mathbf{H}_1 = \mathbf{H} - \mathbf{H}_2,$$

where $\mathbf{H}_2$ evidently has the direction of $\mathbf{M}$. Therefore

$$\mathbf{G} = \mathbf{M} \times \mathbf{H}. \tag{50}$$

Consider a ring of electrons which is rotated from a position where $\mathbf{M}$ is perpendicular to $\mathbf{H}$ to one where $\mathbf{M}$ makes an angle α with $\mathbf{H}$. The potential energy acquired is

$$\begin{aligned} U &= MH\int \sin\alpha\, d\alpha \\ &= -MH\cos\alpha \\ &= -\mathbf{M}\cdot\mathbf{H}. \end{aligned} \tag{51}$$

Therefore if the tendency of the magnetic field to bring all the atomic rings into line is opposed only by the disorganizing effect of thermal agitation, the number of rings per unit volume whose axes make angles with $\overline{\mathbf{H}}$ between α and $\alpha + d\alpha$ is given by

$$\begin{aligned} dN &= \tfrac{1}{2} A e^{-\frac{U}{\frac{3}{2}kT}} \sin\alpha\, d\alpha \\ &= \tfrac{1}{2} A e^{\frac{M\overline{H}\cos\alpha}{\frac{3}{2}kT}} \sin\alpha\, d\alpha, \end{aligned}$$

where T is the absolute temperature and $\frac{1}{2}kT$ the average kinetic energy associated with each degree of freedom.

Integrating, the constant A is found to be given by

$$A = \frac{Nx}{\sinh x},$$

where

$$\begin{aligned} x &\equiv \frac{M\overline{H}}{\frac{3}{2}kT} \\ &= \frac{MB}{\frac{3}{2}kT}. \end{aligned}$$

The magnetic moment per unit volume has the direction of **B** and is given in magnitude by

$$I = \int M \cos \alpha \, dN$$

$$= NM \left\{ \frac{\cosh x}{\sinh x} - \frac{1}{x} \right\}. \tag{52}$$

As **B** increases **I** does also, approaching the saturation value NM for large values of the field. For small fields

$$I \doteq \frac{1}{3} NMx$$

$$= \frac{2}{9} \frac{NM^2}{kT} B, \tag{53}$$

showing that the intensity of magnetization is proportional to the strength of the field. The permeability is given by

$$\mu = 1 + \frac{2}{9} \frac{NM^2}{kT}. \tag{54}$$

The effect under discussion is known as *paramagnetism*, and is characterized by a value of the permeability greater than unity. It is shown to an exceptional degree by iron, in which the effect is given the special name of *ferromagnetism*. The theory given above cannot be considered as more than very roughly approximate to the facts, especially as it gives no explanation of hysteresis, or the lagging of the magnetization behind the field.

40. Energy relations. From (30) and (32) it follows that

$$\mathbf{L} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{L} = -\frac{1}{c} (\mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{L} \cdot \dot{\mathbf{B}}) - \frac{1}{c} (CE^2 + \rho \mathbf{E} \cdot \mathbf{v}).$$

But $$\mathbf{L} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{L} \equiv \nabla \cdot (\mathbf{E} \times \mathbf{L}).$$

Hence

$$\frac{d}{dt} \left\{ \frac{1}{2} (\kappa E^2 + \mu L^2) \right\} + c \nabla \cdot (\mathbf{E} \times \mathbf{L}) + CE^2 + \rho \mathbf{E} \cdot \mathbf{v} = 0.$$

Integrating over any arbitrarily chosen portion of space, and applying Gauss' theorem to the second term,

$$\frac{d}{dt}\int\frac{1}{2}(\kappa E^2+\mu L^2)\,d\tau+c\int(\mathbf{E}\times\mathbf{L})\cdot d\sigma+(CE^2+\rho\mathbf{E}\cdot\mathbf{v})\,d\tau=0. \quad (55)$$

The third term of this expression measures the rate at which work is done by the electromagnetic field on the conduction and convection currents in the medium. Following the line of reasoning pursued in section 28, the conclusion is reached that the first term represents the rate of increase of energy of the field, and the second the flux of energy through the surface enveloping the field. The forms of these expressions suggest that

$$u=\tfrac{1}{2}(\kappa E^2+\mu L^2)$$

is to be considered as the electromagnetic energy per unit volume, and

$$\mathbf{s}=c\,(\mathbf{E}\times\mathbf{L})$$

as the flow of energy per unit cross section per unit time.

41. Metallic conductivity. In developing the electron theory of metallic conduction, the atoms in a metal may be treated as immobile compared with the electrons. Conduction currents of electricity in a metal will be supposed to be due entirely to the drift velocity of the free electrons in the direction of an impressed electric field, and heat conduction will be attributed to the transport of energy by these electrons from one atom to another in the direction of the temperature gradient.

If u is the average velocity of an electron due to thermal agitation, and l the average path described between collisions with atoms, the number of times an electron strikes an atom per second is given by

$$\nu=\frac{u}{l},$$

and equation (8) for the conduction current becomes

$$C\mathbf{E}=\frac{Ne^2l}{2\,mu}\mathbf{E}.$$

As
$$\tfrac{1}{2} mu^2 = \tfrac{3}{2} kT,$$
the electrical conductivity is given by
$$C = \frac{Ne^2lu}{6\,kT}. \qquad (56)$$

In determining the heat conductivity, take the X axis in the direction of the temperature gradient. Consider an electron which is just about to collide with an atom at a distance x from the origin. This atom will have, on the average, a kinetic energy
$$\tfrac{3}{2} kT,$$
but the electron will have the energy
$$\frac{3}{2} k\left(T - \frac{\partial T}{\partial x} l \cos\theta\right)$$
of the last atom with which it collided, where θ is the angle which the electron's path makes with the X axis. During the collision the electron will come into thermal equilibrium with the atom, giving to the latter an amount of energy equal to
$$-\frac{3}{2} k \frac{\partial T}{\partial x} l \cos\theta.$$

Now the number of electrons per unit volume whose paths make angles between θ and $\theta + d\theta$ with the X axis is
$$\tfrac{1}{2} N \sin\theta\, d\theta,$$
and the number of these which pass through unit area at right angles to the X axis in unit time is
$$\tfrac{1}{2} Nu \cos\theta \sin\theta\, d\theta.$$

Hence the flux of energy is
$$-K\frac{\partial T}{\partial x} = -\frac{3}{4} Nluk \frac{\partial T}{\partial x} \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$
$$= -\frac{1}{2} Nluk \frac{\partial T}{\partial x},$$
giving for the thermal conductivity
$$K = \tfrac{1}{2} Nluk. \qquad (57)$$

The ratio of thermal to electrical conductivity,

$$\frac{K}{C} = 3\left(\frac{k}{e}\right)^2 T, \tag{58}$$

varies directly with the absolute temperature and is the same for all metals at a given temperature. Its value for any temperature depends only upon the universal constants k and e, whose values may be determined from experiments having no connection with the metallic conductor in question. The ratio of the conductivities as thus computed is in fair agreement with the ratio as determined directly by experiment.

In obtaining the expressions for the conductivities given above, use has been made of the average velocity of thermal agitation and the average length of path between successive collisions. A more exact calculation gives a slightly different numerical coefficient for the ratio, but one which shows rather worse agreement with the experimental value of this quantity.

42. Reduction of the equations to engineering form. If free charges and currents — either conduction or convection — are present in a material medium, the equations (29) to (32), section 37, of the electromagnetic field may be written in the form

$$\nabla \cdot \mathbf{D} = -\int \nabla \cdot \mathbf{J}\, dt, \quad (59) \qquad \nabla \cdot \mathbf{B} = 0, \quad (61)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (60) \qquad \nabla \times \mathbf{L} = \frac{1}{c}\left(\dot{\mathbf{D}} + \mathbf{J}\right), \quad (62)$$

where $\mathbf{J}$ is the current density; that is, the current per unit cross section. Moreover,

$$\mathbf{J} = C\mathbf{E} + \rho\mathbf{v}, \tag{63}$$

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \mathbf{P} \\ &= \kappa\mathbf{E}, \end{aligned} \tag{64}$$

$$\begin{aligned} \mathbf{B} &= \mathbf{L} + \mathbf{I} \\ &= \mu\mathbf{L}. \end{aligned} \tag{65}$$

The quantities involved in these equations are measured in Heaviside-Lorentz units. If the same quantities as measured

in electromagnetic units are designated by letters with the subscript m, and as measured in electrostatic units by letters with the subscript s, then

$$\rho = c\sqrt{4\pi}\rho_m = \sqrt{4\pi}\rho_s,$$
$$\mathbf{J} = c\sqrt{4\pi}\mathbf{J}_m = \sqrt{4\pi}\mathbf{J}_s,$$
$$\mathbf{P} = c\sqrt{4\pi}\mathbf{P}_m = \sqrt{4\pi}\mathbf{P}_s,$$
$$\mathbf{I} = \sqrt{4\pi}\mathbf{I}_m = c\sqrt{4\pi}\mathbf{I}_s,$$
$$\mathbf{D} = c\sqrt{4\pi}\mathbf{D}_m = \sqrt{4\pi}\mathbf{D}_s,$$
$$\mathbf{B} = \frac{1}{\sqrt{4\pi}}\mathbf{B}_m = \frac{c}{\sqrt{4\pi}}\mathbf{B}_s,$$
$$\mathbf{E} = \frac{1}{c\sqrt{4\pi}}\mathbf{E}_m = \frac{1}{\sqrt{4\pi}}\mathbf{E}_s,$$
$$\mathbf{L} = \frac{1}{\sqrt{4\pi}}\mathbf{L}_m = \frac{1}{c\sqrt{4\pi}}\mathbf{L}_s,$$
$$C = 4\pi c^2 C_m = 4\pi C_s.$$

Therefore, in electromagnetic units the field equations take the form

$$\nabla\cdot\mathbf{D}_m = -\int\nabla\cdot\mathbf{J}_m dt, \quad (66) \qquad \nabla\cdot\mathbf{B}_m = 0, \quad (68)$$

$$\nabla\times\mathbf{E}_m = -\dot{\mathbf{B}}_m, \quad (67) \qquad \nabla\times\mathbf{L}_m = 4\pi(\dot{\mathbf{D}}_m + \mathbf{J}_m), \quad (69)$$

where

$$\mathbf{J}_m = C_m\mathbf{E}_m + \rho_m\mathbf{v}, \tag{70}$$

$$\mathbf{D}_m = \frac{1}{4\pi c^2}\mathbf{E}_m + \mathbf{P}_m$$

$$= \frac{\kappa}{4\pi c^2}\mathbf{E}_m, \tag{71}$$

$$\mathbf{B}_m = \mathbf{L}_m + 4\pi\mathbf{I}_m$$

$$= \mu\mathbf{L}_m. \tag{72}$$

In electrostatic units these equations are

$$\nabla\cdot\mathbf{D}_s = \rho_s', \quad (73) \qquad \nabla\cdot\mathbf{B}_s = 0, \quad (75)$$

$$\nabla\times\mathbf{E}_s = -\dot{\mathbf{B}}_s, \quad (74) \qquad \nabla\times\mathbf{L}_s = 4\pi(\dot{\mathbf{D}}_s + \mathbf{J}_s), \quad (76)$$

where the charge accumulated at any point has been denoted by ρ'_s instead of

$$-\int \nabla \cdot \mathbf{J}_s dt,$$

and
$$\mathbf{J}_s = C_s \mathbf{E}_s + \rho_s \mathbf{v}, \tag{77}$$

$$\mathbf{D}_s = \frac{1}{4\pi}\mathbf{E}_s + \mathbf{P}_s$$

$$= \frac{\kappa}{4\pi}\mathbf{E}_s, \tag{78}$$

$$\mathbf{B}_s = \frac{1}{c^2}\mathbf{L}_s + 4\pi\mathbf{I}_s$$

$$= \frac{\mu}{c^2}\mathbf{L}_s. \tag{79}$$

In practical applications of electrodynamics these equations are generally made use of in integral form. For instance, consider a small charge e permanently at rest at the center of a sphere of radius r. Integrating (73) throughout the region enclosed by the spherical surface,

$$\int \nabla \cdot \mathbf{D}_s d\tau = \int \rho'_s d\tau$$

$$= e_s.$$

But
$$\int \nabla \cdot \mathbf{D}_s d\tau = \int \mathbf{D}_s \cdot d\sigma$$

$$= 4\pi r^2 D_s$$

by Gauss' theorem. Hence

$$D_s = \frac{e_s}{4\pi r^2},$$

and, by virtue of (78),
$$E_s = \frac{e_s}{\kappa r^2}. \tag{80}$$

Again, integrating (67) over a surface,

$$\int \nabla \times \mathbf{E}_m \cdot d\sigma = -\int \dot{\mathbf{B}}_m \cdot d\sigma.$$

But, by Stokes' theorem,

$$\int \nabla \times \mathbf{E}_m \cdot d\sigma = \int \mathbf{E}_m \cdot d\lambda,$$

where the line integral is taken along the boundary of the surface. Therefore

$$\int \mathbf{E}_m \cdot d\lambda = -\int \dot{\mathbf{B}}_m \cdot d\sigma, \tag{81}$$

which is the usual form of Faraday's law of current induction.

Similarly, (69) leads to

$$\int \mathbf{L}_m \cdot d\lambda = 4\pi \int (\dot{\mathbf{D}}_m + \mathbf{J}_m) \cdot d\sigma, \tag{82}$$

which is the form in which Ampère's law is generally expressed.

CHAPTER VIII

ELECTROMAGNETIC WAVES IN MATERIAL MEDIA

43. Isotropic non-conducting media. For wave lengths long compared with atomic dimensions, the electromagnetic field is specified by equations (25) to (28) of the last chapter. For non-conducting media these take the form

$$\nabla\cdot\mathbf{D}=0, \qquad (1) \qquad\qquad \nabla\cdot\mathbf{B}=0, \qquad (3)$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\dot{\mathbf{B}}, \qquad (2) \qquad\qquad \nabla\times\mathbf{L}=\frac{1}{c}\dot{\mathbf{D}}, \qquad (4)$$

where

$$\mathbf{D}=\kappa\mathbf{E},$$

$$\mathbf{B}=\mu\mathbf{L}.$$

Eliminating **B** as in section 29, it is found that

$$\nabla\cdot\nabla\mathbf{E}-\frac{\kappa\mu}{c^2}\ddot{\mathbf{E}}=0, \tag{5}$$

or, eliminating **D**,

$$\nabla\cdot\nabla\mathbf{L}-\frac{\kappa\mu}{c^2}\ddot{\mathbf{L}}=0. \tag{6}$$

These are equations of a wave travelling with velocity

$$V=\frac{c}{\sqrt{\kappa\mu}}. \tag{7}$$

The permeability of all substances is practically unity for frequencies as great as that of light. Hence for light waves it is permissible to write

$$V=\frac{c}{\sqrt{\kappa}}. \tag{8}$$

The *index of refraction* n of a medium is defined as the ratio of the velocity of light *in vacuo* to that in the medium.

Therefore

$$n^2=\kappa, \tag{9}$$

and, as was shown in section 38,

$$\kappa=1+N\frac{e^2}{m}\sum_i\frac{n_i}{k_i^2-\omega^2-2\,i\omega l_i}. \tag{10}$$

In the case of a plane wave it is often convenient to make use of the wave slowness **S** in place of the wave velocity **V**. This quantity is defined as a vector having the direction of the wave velocity but equal in magnitude to its reciprocal. Hence the index of refraction may be defined as the ratio of the wave slowness S in the medium to the wave slowness S_0 *in vacuo*. The electric intensity in the case of a simple harmonic plane wave advancing in a direction making angles α, β, γ with the X, Y, Z axes, may be expressed in terms of the wave slowness by the real part of

$$E = E_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r} - t)},$$

which is far more compact than the equivalent expression

$$E = E_0 e^{i\omega\left(\frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{V} - t\right)}$$

involving the wave velocity.

Equation (8) shows that the wave slowness is given by

$$S = S_0\sqrt{\kappa}, \tag{11}$$

where κ is in general complex. To show the significance of a complex wave slowness, put

$$\mathbf{S} \equiv \mathbf{S}' + i\mathbf{S}'',$$

where $\mathbf{S}'$ and $\mathbf{S}''$ are real. Then

$$E = E_0 e^{-\omega\mathbf{S}''\cdot\mathbf{r}} e^{i\omega(\mathbf{S}'\cdot\mathbf{r} - t)}, \tag{12}$$

showing that the imaginary part of the wave slowness measures the damping of the wave as it progresses into the medium, whereas the real part determines the actual velocity of propagation. The same statement applies to a complex index of refraction. If

$$n \equiv \nu(1 + i\chi),$$

where ν and χ are real, it follows from (10) that

$$\nu^2(1 - \chi^2) = 1 + N\frac{e^2}{m}\sum \frac{n_i(k_i^2 - \omega^2)}{(k_i^2 - \omega^2)^2 + 4\,\omega^2 l_i^2}, \tag{13}$$

$$\nu^2\chi = N\frac{e^2}{m}\sum \frac{n_i\omega l_i}{(k_i^2 - \omega^2)^2 + 4\,\omega^2 l_i^2}. \tag{14}$$

For the portion of the spectrum to which these expressions are to be applied it will be assumed that

$$k_i^2 \gg \omega l_i \gg Nn_i \frac{e^2}{m}.$$

Then

$$\chi \ll 1,$$

and

$$\nu^2 = 1 + N\frac{e^2}{m}\sum \frac{n_i(k_i^2 - \omega^2)}{(k_i^2 - \omega^2)^2 + 4\,\omega^2 l_i^2} \tag{15}$$

approximately. Consider the denominator

$$(k_i^2 - \omega^2)^2 + 4\,\omega^2 l_i^2$$

of one of the terms in the right-hand member of this equation. Except for the region where ω is very nearly equal to k_i the second term of this denominator is negligible compared to the first. Within this region the first is negligible compared to the second. So if ν^2 is plotted against ω^2 in a region extending from ω_1^2 to ω_2^2, such that k_1 lies between ω_1 and ω_2, and k_2, k_3, etc. lie outside this range,

$$\nu^2 = 1 + \frac{Nn_1 \frac{e^2}{m}}{k_1^2 - \omega^2} + \frac{Nn_2 \frac{e^2}{m}}{k_2^2 - \omega^2} + \text{etc.} \tag{16}$$

except in the neighborhood of k_1^2, where

$$\nu^2 = 1 + \frac{Nn_1 \frac{e^2}{m}(k_1^2 - \omega^2)}{4\,\omega^2 l_1^2} + \frac{Nn_2 \frac{e^2}{m}}{k_2^2 - \omega^2} + \text{etc.} \tag{17}$$

Plotting each term separately, the dotted curves of Fig. 17 are obtained. Adding these curves, the full line curve is found to give the relation between ν^2 and ω^2. The portion of the curve from A to B corresponds to regular dispersion, the index of refraction increasing as the frequency becomes greater, whereas the part BC accounts for the anomalous dispersion observed in the neighborhood of an absorption band. It must be remembered that the index of refraction refers always to the *phase velocity* of a train of waves. Hence the fact that this index becomes less than unity on the short wave length side of an

absorption band may not be adduced to show that an electromagnetic signal can be despatched with a velocity greater than that of light *in vacuo.* In fact Sommerfeld has shown that the

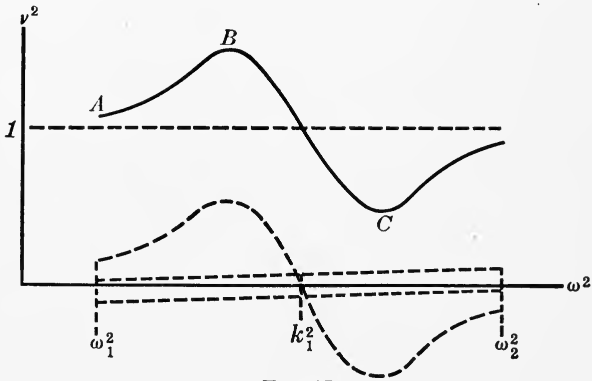

Fig. 17

forerunners of a limited train of waves travel with the same velocity c in all media and are undeviated when they pass from one medium into another.

44. Anisotropic non-conducting media. For wave lengths long compared with atomic dimensions, the first four equations of the electromagnetic field are the same as for isotropic media. The relation between **D** and **E**, however, is different, as the atoms in a body which is not isotropic must be supposed to exert different restoring forces in different directions. Hence equation (36), section 38, for the polarization must be replaced by the more general relation

$$P_x = \epsilon_{11} E_x + \epsilon_{12} E_y + \epsilon_{13} E_z,$$

and similar expressions for P_y and P_z. In the following discussion damping will be assumed negligible. Therefore the coefficients of the components of **E** will be real, though functions of the frequency of the radiation traversing the medium. In vector notation

$$\mathbf{P} = \boldsymbol{\psi} \cdot \mathbf{E}, \tag{18}$$

where $\boldsymbol{\psi}$ is the dyadic

$$\begin{aligned} &\epsilon_{11}\mathbf{ii} + \epsilon_{12}\mathbf{ij} + \epsilon_{13}\mathbf{ik} \\ &+ \epsilon_{21}\mathbf{ji} + \epsilon_{22}\mathbf{jj} + \epsilon_{23}\mathbf{jk} \\ &+ \epsilon_{31}\mathbf{ki} + \epsilon_{32}\mathbf{kj} + \epsilon_{33}\mathbf{kk}. \end{aligned}$$

If u denotes the energy per unit volume of the medium,

$$\begin{aligned} du &= \mathbf{E}\cdot d\mathbf{P} \\ &= (\epsilon_{11}E_x + \epsilon_{21}E_y + \epsilon_{31}E_z)\,dE_x \\ &+ (\epsilon_{12}E_x + \epsilon_{22}E_y + \epsilon_{32}E_z)\,dE_y \\ &+ (\epsilon_{13}E_x + \epsilon_{23}E_y + \epsilon_{33}E_z)\,dE_z. \end{aligned}$$

Now the law of conservation of energy requires that this expression shall be an exact differential. Therefore

$$\epsilon_{ij} = \epsilon_{ji},$$

and the dyadic ψ is self-conjugate. Hence by a proper choice of axes ψ may be put in the form

$$\psi \equiv \epsilon_x\mathbf{ii} + \epsilon_y\mathbf{jj} + \epsilon_z\mathbf{kk}.$$

Now
$$\begin{aligned} \mathbf{D} &\equiv \mathbf{E} + \mathbf{P} \\ &= (1 + \psi\cdot)\,\mathbf{E}. \end{aligned}$$

Put
$$\Phi \equiv \kappa_x\mathbf{ii} + \kappa_y\mathbf{jj} + \kappa_z\mathbf{kk},$$

where
$$\begin{aligned} \kappa_x &\equiv 1 + \epsilon_x, \\ \kappa_y &\equiv 1 + \epsilon_y, \\ \kappa_z &\equiv 1 + \epsilon_z. \end{aligned}$$

Then
$$\mathbf{D} = \Phi\cdot\mathbf{E}, \tag{19}$$

showing that the specific inductive capacity is a dyadic instead of a scalar factor as in the case of an isotropic medium. For **D** and **E** are not in general in the same direction in a medium which is not isotropic. Eliminating **B** and **D** from equations (1), (2), (3), (4), and (19), it is found that

$$\nabla\cdot\nabla\mathbf{E} - \nabla\nabla\cdot\mathbf{E} - \frac{1}{c^2}\Phi\cdot\ddot{\mathbf{E}} = 0. \tag{20}$$

Confining attention to plane waves, the electric intensity and displacement are given by the real parts of

$$\mathbf{E} = \mathbf{E}_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r} - t)},$$

and
$$\mathbf{D} = \mathbf{D}_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r} - t)},$$

respectively.

Therefore
$$d\mathbf{E} = d\mathbf{r}\cdot(i\omega\mathbf{S})\,\mathbf{E} - dt\,i\omega\mathbf{E}.$$

But
$$d\mathbf{E} = d\mathbf{r}\cdot\nabla\mathbf{E} + dt\dot{\mathbf{E}}.$$

Comparing, $$\nabla = i\omega \mathbf{S},$$

and $$\frac{\partial}{\partial t} = - i\omega.$$

Hence (20) becomes

$$S_0^2 \Phi \cdot \mathbf{E} + \mathbf{SS} \cdot \mathbf{E} - \mathbf{S} \cdot \mathbf{SE} = 0. \tag{21}$$

Multiplying by $\mathbf{S}\cdot$, it is seen that

$$\mathbf{S} \cdot \mathbf{D} = 0,$$

or the vector **D** is at right angles to the direction of propagation of the wave. Moreover, equation (21) shows that **D**, **E**, and **S** are in the same plane.

The magnetic intensity may be found from (2). This equation gives

$$\mathbf{S} \times \mathbf{E} = S_0 \mathbf{L},$$

showing that **L** is perpendicular to both **S** and **E**, and hence to **D**. The vectors **D** and **L** lie in the wave front, although **E** makes an angle with this plane. The flux of energy is given by

$$\mathbf{s} = c\mathbf{E} \times \mathbf{L},$$

which is a vector at right angles to **E** in the plane of **D**, **E**, and **S**. Fig. 18 shows the relative directions of the vectors under discussion. Since the flux of energy is not along the wave normal, limited wave fronts will sidestep as they advance, as indicated in the figure. A line drawn in the direction of the flux of energy is known as a *ray*, and the velocity of propagation of energy along this line as the *ray velocity*.

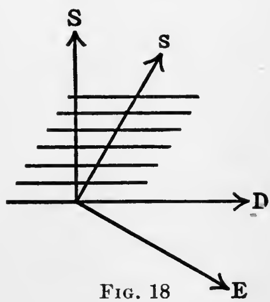

FIG. 18

Consider an infinite number of plane waves passing through the origin at the time 0 in all directions, the vector **D** having all directions in the wave front. The envelope of these plane waves one second later is known as the Fresnel wave surface. To find the equation of this surface, it is necessary to obtain from (21) a relation involving the wave slowness as the only unknown quantity; that is, an

equation between $\mathbf{S}$, $\mathbf{S}_0$, and $\mathbf{\Phi}$ which is true for all possible directions of $\mathbf{E}$. Equation (21) may be written in the form

$$(S_0^2\mathbf{\Phi} + \mathbf{SS} - \mathbf{S}\cdot\mathbf{S}\mathbf{I})\cdot\mathbf{E}_0 = 0, \tag{22}$$

where $\mathbf{I}$ is the idemfactor

$$\mathbf{ii} + \mathbf{jj} + \mathbf{kk}.$$

The dyadic in the parentheses in (22) causes the vector $\mathbf{E}_0$ to vanish. Hence either its antecedents or consequents must be coplanar. This dyadic may be written in the expanded form

$$\begin{aligned}&\{(S_0^2\kappa_x - S^2 - S_x^2)\,\mathbf{i} + S_yS_x\mathbf{j} + S_zS_x\mathbf{k}\}\,\mathbf{i}\\ &+\{S_xS_y\mathbf{i} + (S_0^2\kappa_y - S^2 - S_y^2)\,\mathbf{j} + S_zS_y\mathbf{k}\}\,\mathbf{j}\\ &+\{S_xS_z\mathbf{i} + S_yS_z\mathbf{j} + (S_0^2\kappa_z - S^2 - S_z^2)\,\mathbf{k}\}\,\mathbf{k}.\end{aligned}$$

As the consequents are not coplanar, the antecedents must be. Therefore the scalar triple product of the latter must vanish. Hence

$$\begin{aligned}&(S_0^2\kappa_x - S^2 + S_x^2)(S_0^2\kappa_y - S^2 + S_y^2)(S_0^2\kappa_z - S^2 + S_z^2)\\ &- S_y^2S_z^2(S_0^2\kappa_x - S^2 + S_x^2) - S_z^2S_x^2(S_0^2\kappa_y - S^2 + S_y^2)\\ &- S_x^2S_y^2(S_0^2\kappa_z - S^2 + S_z^2) + 2\,S_x^2S_y^2S_z^2 = 0,\end{aligned}$$

or, reducing,

$$1 + \frac{S_x^2}{S_0^2\kappa_x - S^2} + \frac{S_y^2}{S_0^2\kappa_y - S^2} + \frac{S_z^2}{S_0^2\kappa_z - S^2} = 0.$$

If l, m, n are the direction cosines of the wave normal, this equation becomes

$$\frac{l^2}{V^2 - a^2} + \frac{m^2}{V^2 - b^2} + \frac{n^2}{V^2 - c^2} = 0, \tag{23}$$

where V is the wave velocity, and

$$a^2 \equiv \frac{1}{S_0^2\kappa_x},$$

$$b^2 \equiv \frac{1}{S_0^2\kappa_y},$$

$$c^2 \equiv \frac{1}{S_0^2\kappa_z}.$$

The equation of a plane wave advancing in the direction specified by l, m, n is

$$lx+my+nz=V \tag{24}$$

one second after leaving the origin, where l, m, n satisfy the condition

$$l^2+m^2+n^2=1. \tag{25}$$

The Fresnel wave surface is the envelope of the family of planes obtained by varying l, m, n in equation (24), subject to the conditions specified in (23) and (25). This surface is most easily found by differentiating (23), (24), and (25), and eliminating l, m, n, and V by means of the original equations. The equations obtained by differentiating are

$$x\,dl+y\,dm+z\,dn=dV, \tag{26}$$

$$l\,dl+m\,dm+n\,dn=0, \tag{27}$$

$$\frac{l\,dl}{V^2-a^2}+\frac{m\,dm}{V^2-b^2}+\frac{n\,dn}{V^2-c^2}=kVdV, \tag{28}$$

where

$$k\equiv\frac{l^2}{(V^2-a^2)^2}+\frac{m^2}{(V^2-b^2)^2}+\frac{n^2}{(V^2-c^2)^2}. \tag{29}$$

Let $-p$ and $-q$ be factors by which (27) and (28) respectively may be multiplied so as to eliminate dl and dm when the three equations are added together. As the other differentials are independent of one another, the coefficient of each differential in the sum must vanish, giving

$$\left.\begin{aligned} x&=pl+q\frac{l}{V^2-a^2},\\ y&=pm+q\frac{m}{V^2-b^2},\\ z&=pn+q\frac{n}{V^2-c^2},\end{aligned}\right\} \tag{30}$$

$$1=qkV. \tag{31}$$

Multiplying equations (30) by l, m, n respectively, adding, and making use of the original relations (23), (24), (25), it is found that

$$p=V. \tag{32}$$

Squaring and adding the two sides of equations (30),

$$r^2 = p^2 + q^2k,$$

the product term disappearing on account of (23). Combining with (31) and (32)

$$q = (r^2 - V^2)V. \tag{33}$$

Substituting in (30) the values of p and q just found,

$$x = Vl + \frac{V^2(r^2 - V^2)l}{V^2 - a^2} = \frac{Vl(r^2 - a^2)}{V^2 - a^2},$$

or

$$\frac{x}{r^2 - a^2} = \frac{Vl}{V^2 - a^2} = \frac{x - Vl}{r^2 - V^2},$$

$$\frac{y}{r^2 - b^2} = \frac{Vm}{V^2 - b^2} = \frac{y - Vm}{r^2 - V^2},$$

$$\frac{z}{r^2 - c^2} = \frac{Vn}{V^2 - c^2} = \frac{z - Vn}{r^2 - V^2}.$$

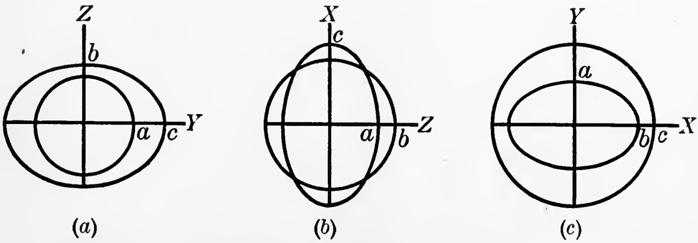

FIG. 19

Multiplying these equations by x, y, z respectively and adding, the required equation

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1 \tag{34}$$

of the Fresnel wave surface is obtained.

In discussing this surface, it will be considered that

$$a^2 < b^2 < c^2.$$

The trace of the surface on the YZ coördinate plane is expressed by

$$(r^2 - a^2)\left(\frac{y^2}{c^2} + \frac{z^2}{b^2} - 1\right) = 0,$$

which is a circle inside an ellipse, as in Fig. 19 (a). On the ZX plane the trace is

$$(r^2 - b^2)\left(\frac{z^2}{a^2} + \frac{x^2}{c^2} - 1\right) = 0,$$

which is a circle cutting an ellipse, as in Fig. 19 (b). Finally, the trace on the XY plane is given by

$$(r^2 - c^2)\left(\frac{x^2}{b^2} + \frac{y^2}{a^2} - 1\right) = 0,$$

which is an ellipse inside a circle, as in Fig. 19 (c).

One octant of the surface is represented in Fig. 20 (a), and a section through the point P is shown in 20 (b). As **D** lies in the plane of the wave normal and the ray, this vector must be

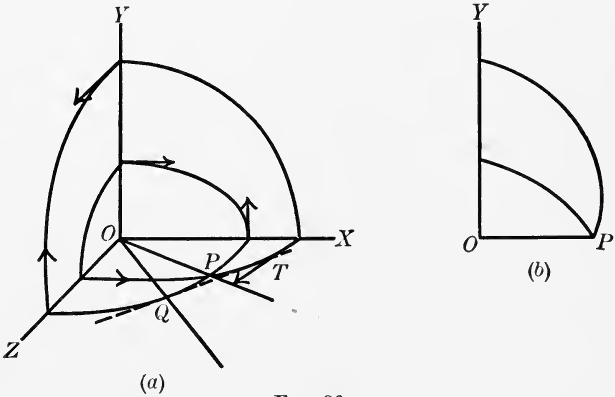

Fig. 20

tangent to each of the elliptical traces, as indicated by arrows in the combined figure. Consequently it must be perpendicular to the planes of the circular traces.

The *primary optic axes* of a crystal are defined as those directions in which the wave velocity is independent of the state of polarization, that is, the direction of **D** in the wave front. Hence the perpendicular $\overline{OQ}$ to the tangent $\overline{QT}$ [Fig. 20 (a)] is one of the primary axes. The other is also in the ZX plane, making an equal angle with $\overline{OX}$ on the other side of the X axis. A crystal which has two optic axes is known as *biaxial*. Obviously, there can be no more than two such axes. Since the constants which determine the intercepts of the Fresnel surface depend upon the three principal specific inductive capacities, which are themselves functions of the wave length, the directions of the optic axes of a biaxial crystal vary with the wave length.

The *secondary optic axes* are defined as those directions in which the ray velocity is independent of the state of polarization. One secondary axis has the direction $\overline{OP}$, the other making an equal angle with $\overline{OX}$ on the other side of the X axis.

A *uniaxial* crystal is one in which two of the quantities a, b, c are equal. If b and c are equal, the crystal is said to be *positive* or *prolate*. The Fresnel wave surface is shown in Fig. 21 (a).

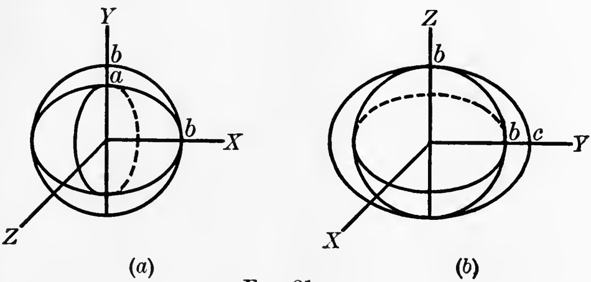

FIG. 21

There is only one axis, and there is no longer any distinction between primary and secondary axes. Moreover, the direction of this axis is independent of the wave length.

If a and b are equal, the crystal is said to be *negative* or *oblate*. The Fresnel wave surface for such a crystal is shown in Fig. 21 (b).

45. Reflection and refraction. Consider a train of plane waves incident at an angle ϕ_1 (Fig. 22) on a plane surface separating two transparent isotropic media. The incident light will be partly reflected and partly transmitted. Let A_1 be the amplitude of the electric vector in the incident radiation, A_1' that in the reflected, and A_2 that in the transmitted radiation. Then the coefficient of reflection R is defined by

$$R \equiv \frac{A_1'}{A_1},$$

and the coefficient of transmission T by

$$T \equiv \frac{A_2}{A_1}.$$

Let the subscripts 1 and 2 refer respectively to the media above and below the plane OY. In the case of the upper medium, letters without primes will refer to the incident light, and letters with primes to the reflected light. Attention will be confined to electromagnetic radiation of wave length long compared to the distances between adjacent molecules of either medium. Hence equations (1) to (4) inclusive specify the field. Moreover, if the media under consideration are transparent, the damping term in expression (10) for the specific inductive capacity is negligible, and this quantity is real.

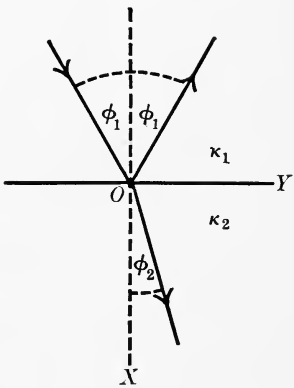

FIG. 22

Choose axes as indicated in the figure, the Z axis extending upward from the plane of the paper. Consider a short pill-box shaped surface, with bases parallel to the YZ plane and axis bisected by this plane. Integrating (1) over the volume enclosed by this surface, and transforming the volume integral into a surface integral by means of Gauss' theorem, it is found that the components of **D** normal to the surface of separation are the same on the two sides of this surface. A similar relation between the normal components of **B** follows from (3).

Consider a rectangle of which one pair of sides is very much longer than the other, so situated that the short sides are perpendicularly bisected by the YZ plane. Integrating (2) over the surface bounded by this rectangle, and transforming the left-hand side of the equation into a line integral by Stokes' theorem, it is found that the components of **E** parallel to the surface of separation are the same on the two sides of this surface, provided $\dot{\mathbf{B}}$ is not infinite at the surface. A similar relation between the parallel components of **L** follows from (4).

Suppose the electric vector in the incident wave to be perpendicular to the plane of incidence; that is, the light is polarized in the plane of incidence. Then the x and y components of the

electric intensity are zero for each wave, and the z components are given by the real parts of

$$\mathbf{E}_1 = \mathbf{k}A_1 e^{i\omega\{S_1(x\cos\phi_1 + y\sin\phi_1) - t\}},$$
$$\mathbf{E}_1' = \mathbf{k}A_1' e^{i\omega\{S_1(-x\cos\phi_1 + y\sin\phi_1) - t\}},$$
$$\mathbf{E}_2 = \mathbf{k}A_2 e^{i\omega\{S_2(x\cos\phi_2 + y\sin\phi_2) - t\}}.$$

Hence, remembering that $n \equiv \dfrac{S}{S_0}$

$$= cS,$$

and that the permeability is unity for the frequencies under consideration, it follows from (2) that

$$\mathbf{L}_1 = \mathbf{i}E_1 n_1 \sin\phi_1 - \mathbf{j}E_1 n_1 \cos\phi_1,$$
$$\mathbf{L}_1' = \mathbf{i}E_1' n_1 \sin\phi_1 + \mathbf{j}E_1' n_1 \cos\phi_1,$$
$$\mathbf{L}_2 = \mathbf{i}E_2 n_2 \sin\phi_2 - \mathbf{j}E_2 n_2 \cos\phi_2,$$

showing that $L_1 = E_1 n_1$, $L_1' = E_1' n_1$, and $L_2 = E_2 n_2$.

Therefore the relations between **E** and **L** on the two sides of the surface of separation lead to the three equations

$$E_1 + E_1' = E_2, \tag{35}$$
$$(E_1 + E_1') n_1 \sin\phi_1 = E_2 n_2 \sin\phi_2, \tag{36}$$
$$(E_1 - E_1') n_1 \cos\phi_1 = E_2 n_2 \cos\phi_2. \tag{37}$$

From (35) and (36) it follows that

$$\frac{\sin\phi_1}{\sin\phi_2} = \frac{n_2}{n_1} = \frac{S_2}{S_1}, \tag{38}$$

the familiar relation attributed to Snell. Dividing (37) by (36),

$$\frac{E_1 - E_1'}{E_1 + E_1'} = \frac{\sin\phi_1 \cos\phi_2}{\sin\phi_2 \cos\phi_1}. \tag{39}$$

Now the exponentials in E_1 and E_1' are the same when x is zero. Hence the electric intensity may be replaced by its amplitude in this equation. Thus

$$\frac{1 - R_\perp}{1 + R_\perp} = \frac{\sin\phi_1 \cos\phi_2}{\sin\phi_2 \cos\phi_1}$$

determines the coefficient of reflection $R_\perp$ for the case where the electric vector is perpendicular to the plane of incidence. Solving,

$$R_\perp = \frac{\sin\phi_2\cos\phi_1 - \sin\phi_1\cos\phi_2}{\sin\phi_2\cos\phi_1 + \sin\phi_1\cos\phi_2} = -\frac{\sin(\phi_1-\phi_2)}{\sin(\phi_1+\phi_2)}. \tag{40}$$

To find the coefficient of transmission $T_\perp$, eliminate E_1' between (35) and (39). Since it follows from (38) that the exponentials in E_1 and E_2 are the same when x is zero, the electric intensity may be replaced by its amplitude, as before. Hence

$$\frac{2-T_\perp}{T_\perp} = \frac{\sin\phi_1\cos\phi_2}{\sin\phi_2\cos\phi_1},$$

whence

$$T_\perp = \frac{2\sin\phi_2\cos\phi_1}{\sin(\phi_1+\phi_2)}. \tag{41}$$

In the case where the electric vector is in the plane of incidence,

$$\mathbf{E}_1 = \{-\mathbf{i}A_1\sin\phi_1 + \mathbf{j}A_1\cos\phi_1\}e^{i\omega\{S_1(x\cos\phi_1+y\sin\phi_1)-t\}},$$
$$\mathbf{E}_1' = \{-\mathbf{i}A_1'\sin\phi_1 - \mathbf{j}A_1'\cos\phi_1\}e^{i\omega\{S_1(-x\cos\phi_1+y\sin\phi_1)-t\}},$$
$$\mathbf{E}_2 = \{-\mathbf{i}A_2\sin\phi_2 + \mathbf{j}A_2\cos\phi_2\}e^{i\omega\{S_2(x\cos\phi_2+y\sin\phi_2)-t\}},$$

and it follows from (2) that

$$\mathbf{L}_1 = \mathbf{k}E_1n_1,$$
$$\mathbf{L}_1' = \mathbf{k}E_1'n_1,$$
$$\mathbf{L}_2 = \mathbf{k}E_2n_2.$$

In this case the relations between **E** and **L** on the two sides of the surface of separation lead to the three equations

$$(E_1+E_1')\,n_1^2\sin\phi_1 = E_2n_2^2\sin\phi_2, \tag{42}$$
$$(E_1-E_1')\cos\phi_1 = E_2\cos\phi_2, \tag{43}$$
$$(E_1+E_1')\,n_1 = E_2n_2, \tag{44}$$

since

$$\kappa = n^2.$$

These equations lead to Snell's law and the following expressions for the coefficient of reflection $R_\parallel$ and coefficient of transmission $T_\parallel$,

$$R_\parallel = \frac{\sin\phi_1\cos\phi_1 - \sin\phi_2\cos\phi_2}{\sin\phi_1\cos\phi_1 + \sin\phi_2\cos\phi_2} = \frac{\tan(\phi_1-\phi_2)}{\tan(\phi_1+\phi_2)}, \tag{45}$$

$$T_\parallel = \frac{2\sin\phi_2\cos\phi_1}{\sin(\phi_1+\phi_2)\cos(\phi_1-\phi_2)}. \tag{46}$$

Examination of the four coefficients of reflection and transmission shows that $R_{\parallel}$ is the only one which can vanish. The *polarizing angle* Φ_1 is defined as the angle of incidence for which this coefficient becomes zero. Therefore

$$\tan(\Phi_1 + \Phi_2) = \infty,$$

and Φ_1 and Φ_2 are complementary. Consequently

$$\Phi_1 = \tan^{-1}\left(\frac{n_2}{n_1}\right).$$

Consider unpolarized light striking a surface at the polarizing angle. The energy associated with the component of the electric vector in the plane of incidence will be entirely transmitted. Consequently the reflected light will consist altogether of radiation in which the electric vector is at right angles to the plane of incidence. Although polarization by reflection should be complete at the polarizing angle, experiment does not show it to be so. This is largely due to imperfect surface conditions.

If n_1 is greater than n_2, the incident radiation may be totally reflected. For convenience the discussion will be restricted to the case for which

$$n_1 \equiv n > 1,$$
$$n_2 = 1.$$

Then, for total reflection,

$$\phi_1 > \sin^{-1}\left(\frac{1}{n}\right),$$

and

$$\sin\phi_2 = n\sin\phi_1$$

is greater than unity, and

$$\cos\phi_2 = i\sqrt{n^2\sin^2\phi_1 - 1}$$

is imaginary. Reference to (40) and (45) shows that both coefficients of reflection will be of the form

$$R = \frac{a - ib}{a + ib} \equiv e^{-i\delta}.$$

Therefore

$$E_1' = A_1 R e^{i\omega\{S_1(0 + y\sin\phi_1) - t\}}$$
$$= A_1 e^{i\omega\{S_1(0 + y\sin\phi_1) - t\} - i\delta},$$

showing that total reflection has produced an alteration in phase without change in amplitude.

Consider the transmitted ray in the case of total reflection.

$$E_2 = A_2 e^{i\omega \{S_2(x\cos\phi_2 + y\sin\phi_2) - t\}},$$

and as $\cos\phi_2$ is imaginary,

$$E_2 = A_2 e^{-\omega\gamma x} e^{i\omega\{S_2 y \sin\phi_2 - t\}},$$

where $$i\gamma \equiv S_2 \cos\phi_2.$$

This is a wave travelling along the surface, with an amplitude which falls off exponentially with x. As the wave does not pass across the surface, no energy is taken from the incident radiation.

46. Rotation of the plane of polarization. Consider a beam of monochromatic plane polarized light travelling through a transparent medium in the direction of an impressed magnetic field **H**. Choose axes so that the X axis is parallel to **H**. Following the method developed in section 38, it is found that equation (9), section 37, leads to

$$\overline{\mathbf{R}} = \frac{\frac{e}{m}}{k^2 - \omega^2}\left\{\overline{\mathbf{E}} - i\frac{\omega}{c}\overline{\mathbf{R}} \times \mathbf{H}\right\},$$

where the term involving the damping constant l in the denominator of the outstanding factor of the right-hand side has been omitted as the medium is transparent.

Solving for $\overline{\mathbf{R}}$,

$$\overline{R}_y = \frac{1}{1 - \frac{\omega^2}{c^2}A^2H^2}\left\{A\overline{E}_y - i\frac{\omega}{c}A^2H\overline{E}_z\right\},$$

$$\overline{R}_z = \frac{1}{1 - \frac{\omega^2}{c^2}A^2H^2}\left\{A\overline{E}_z + i\frac{\omega}{c}A^2H\overline{E}_y\right\},$$

where $$A \equiv \frac{\frac{e}{m}}{k^2 - \omega^2}.$$

Therefore, dropping the strokes,

$$D_y = \kappa E_y - i\frac{\omega}{c}\epsilon AHE_z,$$

$$D_z = \kappa E_z + i\frac{\omega}{c}\epsilon AHE_y,$$

where
$$\kappa \equiv 1 + \frac{NneA}{1 - \frac{\omega^2}{c^2}A^2H^2},$$

$$\epsilon \equiv \frac{NneA}{1 - \frac{\omega^2}{c^2}A^2H^2} = \kappa - 1.$$

Hence, in vector notation

$$\mathbf{D} = \kappa\mathbf{E} - i\frac{\omega}{c}\epsilon A\mathbf{E}\times\mathbf{H}. \tag{47}$$

Since the electric intensity is at right angles to the direction of propagation,

$$\nabla\cdot\mathbf{E} = 0,$$

and elimination of **B** from the field equations (1), (2), (3), (4) gives

$$\nabla\cdot\nabla\mathbf{E} = \frac{1}{c^2}\ddot{\mathbf{D}}, \tag{48}$$

the relation between **D** and **E** being specified by (47).

Now
$$\mathbf{E} = \mathbf{E}_0 e^{i\omega(Sx-t)}.$$

Therefore
$$\nabla\cdot\nabla\mathbf{E} = \frac{\partial^2\mathbf{E}}{\partial x^2}$$

$$= \left\{\frac{\partial^2\mathbf{E}_0}{\partial x^2} - \omega^2S^2\mathbf{E}_0 + 2\,i\omega S\frac{\partial\mathbf{E}_0}{\partial x}\right\}e^{i\omega(Sx-t)},$$

and
$$\frac{1}{c^2}\ddot{\mathbf{D}} = \left\{-\frac{\omega^2\kappa}{c^2}\mathbf{E}_0 + i\frac{\omega^3}{c^3}\epsilon A\mathbf{E}_0\times\mathbf{H}\right\}e^{i\omega(Sx-t)}.$$

Equating real and imaginary terms in these two equal expressions,

$$\frac{\partial^2\mathbf{E}_0}{\partial x^2} = \omega^2(S^2 - \kappa S_0^2)\,\mathbf{E}_0, \tag{49}$$

$$\frac{\partial\mathbf{E}_0}{\partial x} = \omega^2\frac{\epsilon S_0^3A}{2S}\mathbf{E}_0\times\mathbf{H}. \tag{50}$$

The second of these equations shows that if e is positive the plane of polarization rotates in the counter-clockwise sense when viewed from the source of light. Such rotation is called *positive.* Reversing the direction of the applied magnetic field reverses the sense of the rotation. Hence if a beam of plane polarized light is passed through a transparent body along the lines of force of an applied magnetic field, and then reflected and returned over its original path, the rotation of the plane of polarization is not annulled, but doubled. This magnetic rotation was discovered experimentally by Faraday in 1845.

If the angle rotated through is denoted by α,

$$
\begin{aligned}
d\alpha &= \frac{dE_0}{E_0} \\
&= \omega^2 \frac{\epsilon S_0^2 A}{2\, S} H dx, \\
&\doteq \frac{Ne^3}{2\, m^2 c^2 \nu} \frac{\omega^2}{(k^2 - \omega^2)^2} H dx,
\end{aligned}
\qquad (51)
$$

where ν is the index of refraction. Hence the rotation varies with the strength of the magnetic field and the length of the path. In the neighborhood of an absorption band, ω approaches k, and the rotation becomes very large. If the vibrating part of the atom is positively charged, the rotation will be positive when the direction of propagation is the same as that of the magnetic lines of force, while if the vibrating part of the atom is negatively charged, the rotation will be negative. Obviously, only the component of the field in the direction of propagation is effective in producing rotation.

To find the wave slowness, eliminate $\mathbf{E}_0$ from (49) and (50) and solve for S. Thus

$$S^2 - \kappa S_0^2 = -\frac{\omega^2 \epsilon^2 S_0^4 A^2 H^2}{4\, \kappa},$$

or, approximately,

$$S = \sqrt{\kappa} S_0 \left(1 - \frac{\omega^2 \epsilon^2 S_0^2 A^2 H^2}{8\, \kappa^2}\right). \qquad (52)$$

47. Metallic reflection. As a metal contains both free and bound electrons, the equations of the electromagnetic field take the form

$$\nabla\cdot\mathbf{D} = -C\int\nabla\cdot\mathbf{E}dt, \quad (53) \qquad \nabla\cdot\mathbf{B} = 0, \quad (55)$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (54) \qquad \nabla\times\mathbf{L} = \frac{1}{c}(\dot{\mathbf{D}} + C\mathbf{E}), \quad (56)$$

where

$$\mathbf{D} = \kappa\mathbf{E},$$

$$\mathbf{B} = \mu\mathbf{L}.$$

If

$$\mathbf{E} = \mathbf{E}_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r}-t)},$$

then

$$\nabla = i\omega\mathbf{S},$$

$$\frac{\partial}{\partial t} = -i\omega,$$

and integration with respect to the time is equivalent to multiplication by i/ω. Therefore the field equations become

$$\nabla\cdot\left(\kappa + i\frac{C}{\omega}\right)\mathbf{E} = 0, \qquad \nabla\cdot\mathbf{B} = 0,$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \qquad \nabla\times\mathbf{L} = \frac{1}{c}\left(\kappa + i\frac{C}{\omega}\right)\dot{\mathbf{E}}.$$

These are identical with equations (1) to (4), section 43, for a non-conducting medium, provided the specific inductive capacity is replaced by the complex quantity

$$\kappa + i\frac{C}{\omega}.$$

Therefore, remembering that the permeability is unity for light frequencies, the wave slowness is given by

$$S = S_0\sqrt{\kappa + i\frac{C}{\omega}}. \qquad (57)$$

Hence, if

$$\mathbf{S} = \mathbf{S}' + i\mathbf{S}'',$$

then

$$S'^2 - S''^2 = \kappa S_0^2,$$

$$2\mathbf{S}'\cdot\mathbf{S}'' = \frac{C}{\omega}S_0,$$

and

$$\mathbf{E} = \mathbf{E}_0 e^{-\omega\mathbf{S}''\cdot\mathbf{r}} e^{i\omega(\mathbf{S}'\cdot\mathbf{r}-t)}. \qquad (58)$$

Let the YZ plane (Fig. 22, p. 123) be the surface of separation between a region free from matter above this plane and a metallic medium below. Consider a train of plane waves incident on the metallic surface at an angle ϕ_1. Put

$$A \equiv S_0 \sin \phi_1 = S \sin \phi_2,$$
$$C_1 \equiv S_0 \cos \phi_1,$$
$$C_2 \equiv S \cos \phi_2.$$

Then equations (40) and (45), section 45, show that

$$\frac{R_{\parallel}}{R_{\perp}} = \frac{A^2 - C_1 C_2}{A^2 + C_1 C_2}, \tag{59}$$

A and C_1 being real, and C_2 complex. This ratio may be put in the form

$$\frac{R_{\parallel}}{R_{\perp}} = \rho e^{-i\Delta}, \tag{60}$$

where ρ and Δ are real.

Suppose the incident light to be polarized in a plane making an angle of 45° with the plane of incidence. Such radiation may be considered to consist of two trains of waves, one polarized in the plane of incidence and the other at right angles to this plane, which have the same amplitude and are in phase with each other. After reflection these two trains of waves may have different amplitudes, and their phase relation may have been changed in such a way as to produce elliptically polarized light.

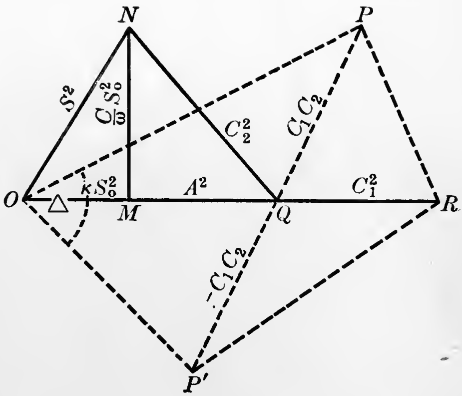

FIG. 23

The ratio of amplitudes after reflection is given by ρ in equation (60), and the difference in phase by Δ. These quantities may be conveniently represented by means of a graph (Fig. 23)

in which real quantities are plotted horizontally and imaginaries vertically. Starting from the origin O, lay off the real component

$$\overline{OM} = \kappa S_0^2$$

of S^2, and the imaginary component

$$\overline{MN} = \frac{C}{\omega} S_0^2.$$

From the same origin lay off the real quantities

$$\overline{OQ} = A^2,$$

$$\overline{QR} = C_1^2,$$

determined by the angle of incidence. Then

$$\overline{QN} = C_2^2,$$

and

$$\overline{QP} = C_1 C_2$$

is laid off by bisecting the angle RQN and making

$$|\overline{QP}| = \sqrt{|C_1^2|\,|C_2^2|}.$$

Connecting O with P and P',

$$\overline{OP'} = A^2 - C_1 C_2,$$

$$\overline{OP} = A^2 + C_1 C_2.$$

Hence the ratio of amplitudes after reflection is given by

$$\rho = \frac{|\overline{OP'}|}{|\overline{OP}|}, \tag{61}$$

and the difference in phase between the two components by

$$\Delta = \angle P'OP. \tag{62}$$

48. Zeeman effect. Consider an electron which may vibrate under the influence of a force of restitution proportional to the electron's displacement from its position of equilibrium inside the atom. The equation of motion of such an electron has the components

$$m\frac{d^2x}{dt^2} = -kx,$$

$$m\frac{d^2y}{dt^2} = -ky,$$

$$m\frac{d^2z}{dt^2} = -kz,$$

showing that its natural vibration has a frequency defined by

$$\omega = \sqrt{\frac{k}{m}}.$$

In the presence of a magnetic field **H** parallel to the Z axis the electron under consideration is subject to an additional force

$$\frac{e}{c}\mathbf{v} \times \mathbf{H},$$

so that the components of the equation of motion become

$$m\frac{d^2x}{dt^2} = -kx + \frac{eH_z}{c}\frac{dy}{dt},$$

$$m\frac{d^2y}{dt^2} = -ky - \frac{eH_z}{c}\frac{dx}{dt},$$

$$m\frac{d^2z}{dt^2} = -kz.$$

Solving these equations, it is found that

$$x = A_1 \cos(\omega_1 t + \delta_1) \quad \text{and} \quad y = -A_1 \sin(\omega_1 t + \delta_1), \tag{63}$$

or $$x = A_2 \cos(\omega_2 t + \delta_2) \quad \text{and} \quad y = A_2 \sin(\omega_2 t + \delta_2), \tag{64}$$

$$z = A_3 \cos(\omega t + \delta), \tag{65}$$

where $$\omega_1^2 - \frac{e\omega_1}{mc} H_z = \omega^2,$$

or, approximately, $$\omega_1 \doteq \omega + \frac{e}{2\,mc} H_z; \tag{66}$$

and $$\omega_2^2 + \frac{e\omega_2}{mc} H_z = \omega^2,$$

or $$\omega_2 \doteq \omega - \frac{e}{2\,mc} H_z. \tag{67}$$

Equations (63) and (64) represent rotation in circles in the XY plane in the negative and positive senses respectively relative to the Z axis. The effect of the magnetic field is merely to change the central force from

$$kr$$

to $$kr \mp e\beta H_z.$$

Consider a body which emits light in consequence of the vibrations of electrons which are held in the atoms by simple harmonic forces of the type under discussion. Suppose a magnetic field to be applied in the direction of the Z axis, and let the source of light be viewed along the X axis. Vibrations in the X direction will emit no radiation in the direction from which the light is being observed. Vibrations in the Y direction will give rise to light polarized with the electric vector parallel to the Y axis of frequencies ω_1 and ω_2, while vibrations in the Z direction will produce light polarized with the electric vector parallel to the Z axis of frequency ω. Therefore when a source of light in a magnetic field is viewed in a direction at right angles to the lines of force, each spectral line will be resolved into three components. The central undisplaced component will be polarized with the electric vector parallel to the field, and the two displaced components with the electric vector at right angles to the field.

If the source of light is viewed along the Z axis, no radiation will reach the observer due to vibrations parallel to this axis. Vibrations perpendicular to the Z axis will give rise to circularly polarized light of frequencies ω_1 and ω_2. Consequently, when a source of light in a magnetic field is viewed along the lines of force, each line will be resolved into two components, circularly polarized in opposite senses and equally displaced on either side of the original line. There will be no undisplaced component. The sense of the circular polarization of the two displaced components depends upon the sign of the vibrating electrons, which are thus shown to be negative. The ratio of charge to mass of the negative electron may be calculated from the displacements observed. This method was one of the earliest employed to obtain the numerical value of this important constant.

While the results obtained from theory are entirely confirmed by experiment in many cases, a large number of lines are split up into more than three components by a magnetic field. It is believed that these are compound lines, which the optical apparatus employed is not powerful enough to resolve.

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